

Researches in Acoustics.

By

C. V. RAMAN, M.A.,

*Professor-Elect in the Sir Taraknath Palit Chair of Research in Physics,
University of Calcutta ;*

*Life-Member of the Indian Association for the Cultivation of Science ;
and*

Director, Astronomical Society of India.

CALCUTTA :

Published by the Baptist Mission Press.

1914.

PREFATORY NOTE.

THIS Volume brings together a selection from my published papers in Acoustics. Each Section marks a definite advance on the existing knowledge of the subject, and the whole forms a record of a connected and systematic body of research carried out in the intervals of my official duties as a Gazetted Officer of the Indian Finance Department. My papers in Optics and on the Theory of Elastic Impact will form similar and separate Volumes.

C. V. RAMAN.

EXPERIMENTAL INVESTIGATIONS ON THE MAINTENANCE OF VIBRATIONS.

By C. V. Raman, M.A.

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PREFATORY NOTE.

These Investigations formed the subject of Lectures delivered at special meetings of the Indian Association for the Cultivation of Science on the 9th January, 1909, with Sir Gooroodas Banerjee presiding, and on the 9th May, 1912, with the Hon'ble Justice Sir Asutosh Mookerjee, F.R.S.E., etc. in the chair.

I. ON A NEW FORM OF MELDE'S EXPERIMENT.

This will be found described in my notes in 'Nature' of the 4th November, 1909, and in the 'Physical Review' of March, 1911 (Bulletins of the Indian Association Nos. 2 and 3). When properly performed its results easily surpass in beauty and interest, those obtained with the usual arrangements in Melde's experiments. The modified form of the experiment was devised in the course of

my work of 1906 at the Presidency College, Madras, when endeavouring to clear up certain anomalous observations by Mr. V. Appa Rao. The fine cotton or silk string which is maintained in vibration is attached to the prong of a tuning-fork which is best maintained electrically (though indeed a bowed fork is suitable enough) and is held so that it lies in a plane perpendicular to the prongs but in a direction *inclined* to their line of vibration.

Under these circumstances the motion of the prong may be resolved into two components, one parallel and the other perpendicular to the string. The latter transverse component maintains an oscillation having the same frequency as that of the fork when the tension of the string is suitably adjusted. The length of the string should be such that under the action of this force the string divides up into an even number of ventral segments, say two. The first, i.e. longitudinal component of the obligatory motion, will then generally be found to maintain simultaneously an oscillation having half the frequency of that of the fork. The success of the experiment lies in isolating the two vibrations, the frequency of one of which is double that of the other, into perpendicular planes. This is easily secured by a simple little device. The end of the string is attached to a loop of thread which is passed over the prong, instead of directly to the prong itself. The result of this mode of attachment is that the frequencies of vibration in the two planes at right angles differ slightly and this has the desired effect of keeping the two component vibrations confined to their respective planes, if the tension of the string lies anywhere within a definite range.

With the arrangements described above it is evident that the motion of each point on the string in a plane transverse to the length should be one of the Lissajous figures for the interval of the octave. The shape of this figure depends on the phase-relation between the component oscillations, and this is determined by the precise value of the tension of the string. It is well to state at once that when the tension is somewhat in excess the curves are parabolic arcs. Plate I shows a stereo-photograph of a string maintained in an oscillation of this type.

We proceed to discuss the approximate theory of this case. The motion of the prong of the tuning-fork may be put equal to $y \cos pt$.

The component of this transverse to the string may be put equal to $\gamma \cos pt \sin \theta$. If the distance of any point on the string from the fixed end when at rest is x , the transverse components of the maintained motion may to a first approximation be written as under.

$$Y = \gamma \sin \theta \frac{R_x}{R_b} \cos \left(pt + E_x - E_b \right) \quad (1)$$

Vide Lord Rayleigh's 'Theory of Sound,' Art. 134.

$$Z = B \cos \left(\frac{pt}{2} + E \right) \sin \frac{\pi x}{b} \quad (2)$$

If we exclude any consideration of the motion at points near the nodes of the maintained oscillation, equation (1) may be written in the simple form

$$Y = \gamma \sin \theta \sin \frac{px}{a} \cos \left(pt + E' \right).$$

If $\frac{p}{a} = \frac{2\pi}{b}$, (1) and (2) may be written in the form—

$$Y = A \sin \frac{2\pi x}{b} \cos \left(pt + E' \right) \quad (3)$$

$$Z = B \sin \frac{\pi x}{b} \cos \left(\frac{pt}{2} + E \right) \quad (4)$$

It should be understood that in these equations Y and Z do *not* refer to the co-ordinates of any point fixed relatively to the string but to the points at which a plane transverse to its equilibrium position cuts the surface generated by the moving string. The distinction is of importance in view of the fact that each point on the string possesses a small longitudinal motion derived from that imposed by the fork and the x co-ordinate of any point fixed relatively to the string is therefore not itself constant.

In particular cases equations (3) and (4) may be reduced to very simple forms. Thus if $E' = 2E$ and also in the special case when both E and E' are equal to zero—

$$\frac{Y}{A} \operatorname{cosec} \frac{2\pi x}{b} = \frac{2Z^2}{B^2} \operatorname{cosec}^2 \frac{\pi x}{b} - 1 \quad (5)$$

which is the equation of the surface generated by the moving string, the sections of which by planes perpendicular to the axis

of x are parabolic arcs. The curvature of these parabolic arcs lies in opposite directions for values of x less and greater than $b/2$. This is exactly the type of vibration shown in the stereo-photograph (Plate I). It will be noticed that in the half of the string near the tuning fork the curvature of the parabolic arcs is in one direction and in the other half in the opposite direction.

At the mid-point there is practically no transverse motion in one of the perpendicular planes. This is also evident in the Plate, but no photograph can give a really adequate idea of the beauty of the stationary form of vibration, which must be seen to be fully appreciated.

It is not difficult to make out from general considerations why when the tension is somewhat in excess, the phase-relation between the component vibrations is such as to give us a parabolic type of vibration. It is clear that when the free period of vibration of the half-length of the string is somewhat less than that of the fork, the phase of the oscillation maintained by the transverse obligatory motion is very approximately in agreement with that of the obligatory motion itself, i.e. $E' = 0$. Under the same circumstances and generally also whenever a large amplitude of vibration is maintained by the longitudinal component of the motion of the fork, the phase of the oscillation of half-frequency is such that the displacement is very nearly a maximum when the tension is a minimum, and vice versa. This is what it would be if $E = 0$. Since E and E' are both zero, equation (5) gives us the required type of vibration.

A parabolic type of motion should also be obtained when $E' = 2E + \pi$. The equation of the surface generated by the moving string in this case may be obtained by merely writing $-Y$ for Y in equation (5)

$$\frac{Y}{A} \operatorname{cosec} \frac{2\pi x}{b} = 1 - \frac{2Z^2}{B^2} \operatorname{cosec}^2 \frac{\pi x}{b} \quad (6)$$

The sections of this surface by planes normal to the axis of x are parabolic arcs, but it will be noticed that their curvatures are in the opposite direction to those given by equation (5). A rough approximation to the case given by equation (6) is obtained when the tension of the string is somewhat in defect, i.e. the free period

of the half-length of the string is more than the period of the fork, and the longitudinal component of the vibration of the fork maintains an oscillation of large amplitude.

It remains now to consider the intermediate case where $E' = 2E + \pi/2$. It is evident *a priori* that in this case the sections of the surface described by the moving string by planes normal to the axis of x should be 8 curves, and this is readily verified by experiment. The tension of the string, to obtain a motion of this type, should be roughly that at which the transverse obligatory motion maintains the most vigorous vibration, and a large motion is also maintained by the longitudinal component.

This leads me on to consider a very interesting point which was referred to above in passing. From equation (3) it appears that when $x = b/2$, $Y = 0$. This point is the 'node' of the oscillation maintained by the transverse obligatory motion. Strictly speaking Y is not zero at this point. There is a very appreciable small motion at the node, the magnitude of which is given by equation (1). This equation may by changing the origin of time be written in the more intelligible form—

$$Y = \frac{\gamma \sin \theta}{R_b} \left(\sin ax \cos pT + \frac{kx}{2a} \cos ax \sin pT \right) \quad (7)$$

At the node the first term within the bracket is zero but the second term remains finite. It will be seen that the phase of the second term differs from that of the first by quarter of an oscillation. When the sections of the surface generated by the vibrating string at other points are 8 curves, as described in the preceding paragraph, the section of the surface at the node itself by a plane normal to the string is a parabolic arc with a fairly large radius of curvature. This is readily verifiable by experiment. I here refer to the motion in a plane transverse to the string and this is quite distinct from the curvature due to the small motion parallel to the axis of x which each point of the string (other than the fixed end) possesses in virtue of the longitudinal motion imposed by the tuning-fork.

Equations (3) and (4) represent the curves along which the string lies at any given instant. They are of course not plane curves at all (except at the epochs when either Y or Z or both are

everywhere zero) and are exceedingly pretty. With the parabolic type of oscillation the peripheral curve, i.e. the position of the string at its extreme outward swing, can readily be seen. In other cases, intermittent light is required to render these curves visible. Probably the most satisfactory arrangement is to use intermittent illumination having a frequency double that of the tuning fork which maintains the string in vibration, so that four views are obtained simultaneously and by their disposition give a much more vivid idea of the mode of motion than would be had if only one or two positions of the string were visible. For work of this kind, a stroboscopic disk with narrow radial slits and run by a Rayleigh motor synchronous with the driving tuning-fork is an extremely useful piece of apparatus. The motor purchased by the Association has thirty teeth on its armature-wheel and I have had two stroboscopic disks made with thirty and sixty slits respectively, either of which can be mounted on the motor. It is not necessary to hold the eye close to the stroboscopic disk for many purposes. If the disk is vertically held and the vibrating string is horizontal and parallel to the disk and is observed through the top row of slits, i.e. through those moving in a direction parallel to the string, the latter is seen as if divided up into a fairly large number of ventral segments. This effect is due to the fact that the string is observed through different parts of the revolving disk and it is therefore seen in successive cycles of phase along its length. We get practically a series of replicas of the string with the same amplitude of motion but of greatly diminished length, i.e. with magnified curvature. This is specially advantageous in the present case. The vibrating string should be brightly illuminated when under observation through the stroboscopic disk.

In concluding this section I must remark that some phenomena of interest are observed when the two modes of vibration, the frequency of one of which is double that of the other, are *not* isolated in perpendicular planes. When not kept in check by some device of the kind described at the commencement of this note, the oscillation of higher frequency has a tendency to settle down into circular or elliptical motion and some very curious types of vibration are obtained by its composition with the plane vibration of half frequency. The form of these types can be varied by

altering the inclination of the string and its tension. We need not however pause to discuss them in detail. Interesting as some of these modes of motion are, they sink into insignificance when compared with some of the compound types of vibration maintained by a simple harmonic force that will be illustrated in Section V of this paper.

II. THE SMALL MOTION AT THE NODES OF A VIBRATING STRING.

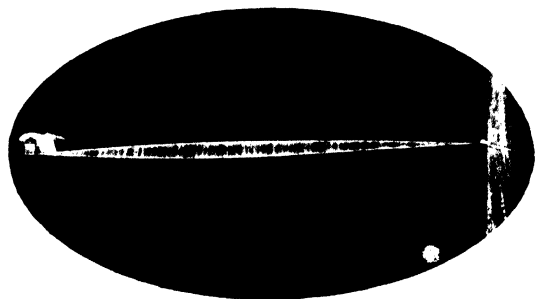
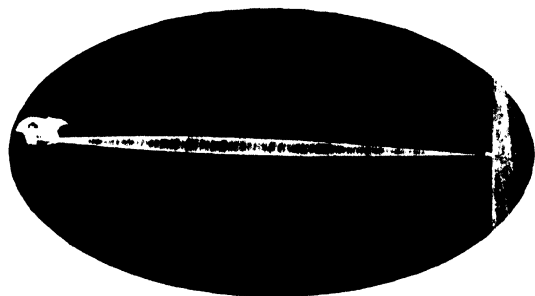
I drew attention in recent publications (quoted at the commencement of the previous section) to some remarkable features of the small motion at the nodes of a vibrating string which it appears had not previously been noticed. A vivid idea of the types of motion obtaining can as I showed be had by observing under periodic illumination of approximately double the frequency of that of the oscillation. I have since succeeded in obtaining photographs under actual experimental conditions of the appearances observed.

When a stretched string is maintained in oscillation in segments by a periodic force or an obligatory motion imposed transversely at one point on it, the nodes are not of course points of absolute rest, as the energy requisite for the maintenance of the motion is transmitted through these points. Certainly the best way of observing what exactly takes place at the nodes is to use intermittent illumination, the frequency of this being nearly double that of the vibrations. We would then see *two* slowly moving positions of the string which obviously represent opposite phases of the actual motion. If the nodes were points of absolute rest, then these two positions would intersect at fixed points. One naturally expects that the 'nodes' or points of intersection actually seen under the intermittent illumination should never depart very far from the positions of the real nodes of the oscillation, i.e. the positions where the string is seen under non-intermittent illumination to divide up into segments. But this is not the case. The 'nodes' seen under the intermittent illumination travel along the string over an extraordinary range, in fact over a distance equal to the whole length of a loop. This striking effect is very readily observed, any device for securing intermittent

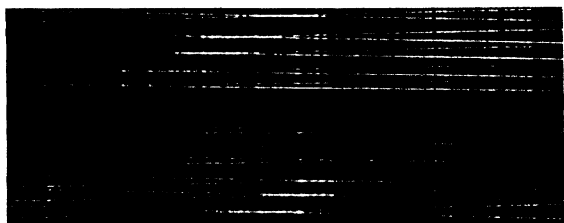
illumination of approximately the correct frequency being sufficient for the purpose.

While visual observation is quite simple, much the most satisfactory arrangement both for ordinary daylight observation and for photographic work is the use of a stroboscopic disk with radial slits mounted on a Rayleigh motor which is actuated by the intermittent current from a self-maintaining tuning-fork interrupter and therefore runs synchronously with it. The tuning-fork interrupter, which is of frequency 60 per second, maintains the string in transverse oscillation of the same frequency. It also drives the synchronous motor on which is mounted a stroboscopic disk having just double as many apertures as the armature-wheel has teeth. The disk therefore gives two views of the fork and of the string maintained by it, which are practically stationary provided the point of observation is fixed and the motor is running satisfactorily. By changing the point of observation (which should be so chosen that the radial slits are parallel to the string and move at right angles to it) the successive stages of the motion and of the travel of the 'nodes' can be observed at leisure.

For photographic work, the stroboscopic disk is held vertically and the camera employed is brought close behind that one of the rectangular slits on the disk which is horizontal. The lens is stopped down by a plate which has a rectangular slit cut in it to correspond with those on the disk. The string and the aperture on the lens of the camera are both horizontal and by racking up the lens-front by successive small distances till it has moved through a length equal to that between contiguous apertures on the disk, a complete set of photographs can be obtained on one plate showing the successive stages of the motion of the string. Plate II reproduces a photograph obtained in this manner, the centre of the field being the position of the node of the vibrating string as seen by non-intermittent light. It shows the cycle of changes in 13 stages and was obtained by moving up the lens-front through very small distances each time. It will be seen that the point of intersection or 'node' which is first in the centre moves off to one side of the field, first slowly and then more quickly, till after the lapse of a time which can be seen to be exactly half the period of the cycle it has gone well off the plate and the two positions of



A NEW FORM OF WELDS EXPERIMENT.



STROBOSCOPIC PHOTOGRAPH
OF THE SMALL MOTION
AT THE NODE OF A VIBRATING STRING

the string seen in the photograph are sensibly parallel. Direct observation shows that the point has moved off to a distance equal to half the length of a segment. It simultaneously appears at an equal distance on the other side and moves in from that direction first quickly and then more slowly till it reaches the centre again, and the cycle is complete.

The explanation of these phenomena is that the small motion at the node is not in the same phase as the large motion elsewhere. It is evident from the photograph that the small displacement at the node is a maximum when the large motion elsewhere is a minimum: in other words that its phase differs by exactly quarter of a period of the vibration from that of the general motion of the string. An independent method of demonstrating this was discussed incidentally in Section I above. Equation (7) of that section contains in a nutshell the complete theory of the case. The sign of the phase of the small motion at any node may be found from the following rule, which is verified by observation. If the tuning-fork which imposes the obligatory transverse motion is exactly at a node, it is opposite in phase to the small motion at the next node, and in the same phase as the motion at the node next after that, and so on.

III. THE AMPLITUDE AND THE PHASE OF OSCILLATIONS MAINTAINED BY FORCES OF DOUBLE FREQUENCY.

In a note published in 'Nature' of the 9th December, 1909, and again more fully in a communication under the title "Remarks on a paper by J. S. Stokes on 'Some curious phenomena observed in connection with Melde's Experiment,'" published in the 'Physical Review' for March, 1911 (see Bulletins 2 and 3 of the Association), I drew attention to the fact that there were considerable discrepancies between the facts of observation and the theory first published by Lord Rayleigh (*Phil. Mag.*, April 1883, August 1887, and 'Theory of Sound,' Art. 68*b*) as regards the maintenance of vibrations by forces of double frequency, and I also indicated the cause of these discrepancies. The phenomena observed are not only interesting in themselves but are very important in connection with the general theory of the maintenance of vibrations by a variable spring which I shall discuss in the succeeding Sections.

Lord Rayleigh starts with the following as his equations of motion :—

$$\ddot{u} + k\dot{u} + (n^2 - 2a \sin 2pt) u = 0 \quad (1)$$

and assuming that u may be put equal to

$$A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt + \&c. \quad (2)$$

proceeds to find the conditions that must be satisfied for the assumed state of steady motion to be possible. This he does by substituting (2) for u in the left-hand side of equation (1) and equating to zero the coefficients of $\sin pt$, $\cos pt$, etc. The relations thus obtained are (to a first approximation)

$$\frac{B_1}{A_1} = \frac{\sqrt{(a - kp)}}{\sqrt{(a + kp)}} = \tan E \quad (2)$$

$$(n^2 - p^2)^2 = a^2 - k^2 p^2 \quad (3)$$

By a trigonometrical transformation equation (2) may be written in the form

$$kp = a \cos 2E. \quad (4)$$

These equations show that the phase of the motion is independent of the amplitude maintained and that the latter quantity is indeterminate.

It is possible to test experimentally the phase-relation as given by equations (2) and (3). The oscillatory system used for this purpose is a stretched string which is maintained in vibration by a periodic variation of tension of double frequency imposed on it with the aid of a tuning-fork. In this case the term $-2a \sin 2pt$ is proportional to the motion of the tuning-fork and u corresponds to the maintained vibration of the string. The experimental problem therefore reduces itself to a determination of the phase-relation between the vibrations of the fork and the string, the frequency of one of which is double that of the other. This can be investigated by two distinct devices. The first is

Mechanical composition of the two motions. This is automatically effected and needs no special experimental arrangements. For, each point on the string (except the fixed end) has two motions at right angles to each other. The first is transverse to the string and is merely that due to its general vibration. The

second is longitudinal to the string and is due to the motion in that direction of the prong of the fork to which the string is attached. The resulting path of any point on the string lies in the plane of oscillation and is one of the Lissajous figures for the interval of the octave. This curve may easily be rendered conspicuous by attaching a small fragment of a silvered bead to a point on the string near the tuning-fork. This is the most convenient position, though in case the vibration of the string is in two or more ventral segments, the bead may also be attached near any one of the other nodes as well. The second method is

Optical composition of the two motions, and this is undoubtedly the more elegant of the two. To effect this, a small mirror is attached to the extremity of the prong of the fork. A tuning-fork with a steel mirror fixed to the end of one prong (see Lord Rayleigh, 'Theory of Sound,' Art. 39) may well be used for the purpose. One point on the stretched string is illuminated by a transverse sheet of light from a lantern or with sunlight and a cylindrical lens. When the string is set in vibration, this appears drawn out into a luminous straight line which is viewed by reflexion first at a fixed mirror and then at the oscillating mirror attached to the tuning-fork. If the plane of vibration of the prongs is at right angles to that in which the string vibrates (this may be secured by a simple experimental device), the illuminated point is seen to describe a Lissajous figure which renders evident at once the phase-relation under investigation.

Working by either of these methods, it is found that the phase of the motion is *not* independent of the amplitude maintained with any given initial tension. The best way of showing this is to use a bowed fork and after starting the motion with a large amplitude to gradually allow it to die away, the Lissajous figure or 'Curve of motion' as I shall call it and the changes that occur in it being watched during the process. It is observed that the initial curve of motion and the alterations that it undergoes when the motion is gradually damped down, both depend on the initial tension of the string. With a high initial tension so that the string can be maintained in its fundamental mode of vibration only by vigorous bowing of the fork, it is found that the curve is a parabolic arc which is convex to the tuning-fork and remains

as such when the motion dies away. This state of matters continues so long as the initial tension is considerably in excess of that at which the free period of vibration of the string for small amplitudes is equal to the period of the fork. As the tension is gradually reduced, it will be observed while the initial curve for large amplitudes is a parabolic arc, it becomes modified into a looped figure as the amplitude decreases, still however remaining convex. When the tension is still further reduced so that the free period of the string for small oscillations is equal to that of the fork, the curve of motion for large amplitudes is still approximately parabolic or at any rate a looped figure convex to the fork, but as the motion dies away it alters into an 8-shaped figure. The most remarkable changes are however observed with a still smaller tension. In this case very large amplitudes of motion are maintained and the initial curve of motion is still convex, but as the motion is damped away it becomes an 8-shaped figure and finally a looped figure *concave* to the fork. At this stage the motion suffers very rapid damping, and when the initial tension is below a certain value a minimum amplitude of motion of the string exists below which steady motion is not possible. In the final stage with the smallest amplitudes, the curve of motion is a parabolic arc with its *concavity* towards the fork.

To enable these observations to be satisfactorily explained, it is necessary to modify Lord Rayleigh's theory so as to take into account the variations of tension that exist in free oscillations of sensible amplitude and are proportional to the square of the motion. In my paper on 'Photographs of Vibration Curves' in the *Phil. Mag.* for May 1911 (see Bulletin No. 5 of the Indian Association) I showed experimentally that such variations of tension exist by causing them to act on a sounding-board, which was held normal to the wire and would therefore have otherwise remained appreciably at rest. The vibration curve of the sounding-board was photographed on a moving plate along with and immediately above the vibration curve of the wire itself. It was observed that the frequency of the vibrations of the sounding-board was generally double that of the oscillations of the wire. But when the equilibrium position of the wire was a catenary of small curvature and its oscillations took place in a vertical plane, the

motion of the sounding-board excited by them had a component of frequency identical with their own. These and other observations proved conclusively that variations of tension existed in free oscillations of sensible amplitude which were due to the second order differences in length between the equilibrium and displaced positions of the wire or string and were in fact proportional to the square of the displacement. Taking these into account the modified equation of motion under the action of forces varying the spring may be written as

$$\ddot{u} + k\dot{u} + (n^2 - 2a \sin 2pt + \beta u^2) u = 0. \quad (5)$$

Assuming that

$$u = A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt + \&c.$$

and substituting in the left-hand side of equation (5), we obtain the conditions that must be satisfied for steady motion to be possible by equating to zero the coefficients of $\sin pt$, $\cos pt$, etc. Neglecting the quantities A_3 , B_3 , etc. as too small appreciably to effect the final result, we obtain

$$\tan E = \frac{B_1}{A_1} = \frac{a - kp}{n^2 - p^2 + F} \quad (6)$$

$$(n^2 - p^2 + F)^2 = a^2 - k^2 p^2 \quad (7)$$

where

$$F \text{ is equal to } \frac{3\beta}{4} (A_1^2 + B_1^2)$$

and is therefore proportional to the square of the amplitude of motion. Equation (6) may as before be written in the form

$$kp = a \cos 2E \quad (8)$$

From these equations we may draw the following inferences:

If $a < kp$ no steady motion is possible. When

$$n < p \text{ and } (p^2 - n^2)^2 > a^2 - k^2 p^2$$

maintenance would evidently be impossible unless F had a certain finite minimum value. This is in accordance with the results of experiment. When the initial amplitude is less than that given by this minimum, the motion cannot be sustained and rapidly dies away. On the other hand, if the initial amplitude is equal to or

greater than the required minimum, it shows a marked tendency to increase rapidly of itself up to many times the initial value. The reason for this is pretty clear from equation (7). When $(p^2 - n^2)$ is positive and greater than the right-hand side of that equation, any increase of the amplitude of motion *diminishes* the quantity on the left, with the result that a still further increase in amplitude is entailed, and it continues to increase till $(n^2 - p^2 + F)$ again becomes positive in sign and equal to $(\alpha^2 - k^2 p^2)^{\frac{1}{2}}$ in magnitude. A somewhat similar rapid increase in the amplitude (though not of such a marked character) takes place in all cases where the initial tension is less than the theoretical value. The motion is however capable of starting from infinitely small vibrations if $(p^2 - n^2)^2$ is equal to or greater than $(\alpha^2 - k^2 p^2)$. On the other hand, when the tension is sufficient or in excess no such phenomenon is observed. The increase of the motion from infinitely small amplitudes up to the value required to satisfy equation (7) is then quite gradual.

Again it is evident that for given values of α and kp , F is not a maximum when the tension is equal to the theoretical value. In other words, the maximum amplitude of motion is *not* obtained when the free period of the string for small oscillations is double that of the tuning-fork. This somewhat paradoxical result is entirely verified by observation. In fact it is clear that the amplitude maintained is largest when n is less than p and has as small a value as is consistent with steady motion in the given mode, in other words when the initial tension of the string is considerably in defect.

We now proceed to discuss the phase of the maintained motion. This is given by equations (6) or (8) above. From the former it is evident that E , i.e. the phase-difference, is always positive if $(n^2 - p^2 + F)$ is of that sign. The curve of motion is therefore convex to the fork when the tension is in excess and also when it is in defect, provided the amplitude of motion is sufficiently large. The maximum positive phase-difference is $\pi/4$ and this is attained when α is large compared with kp . It can be seen from equation (7) that a large variation of tension is required to start the motion when the initial tension is high. The curve of motion is then a parabolic arc convex to the fork and continues

as such so long as the tension is in excess and the amplitude is sufficiently large. But with small amplitudes, the phase-difference though positive is less than $\pi/4$ and the curve of motion is a looped figure. When the initial tension is equal to the theoretical value and the amplitude of motion is very small $a = kp$ and $E = 0$ and the curve of motion is shaped like an 8. This is in agreement with observation. But when the amplitude increases, the phase-difference again becomes finite and the curve is convex to the fork. When the initial tension of the string is in defect, the phase-difference is positive or negative according as the amplitude is large or small and the curve of motion is convex or concave under the respective circumstances. It is not at all difficult to observe any of these different cases, though in order to maintain the motion steadily with the curve in the concave position some careful adjustment of the amplitude of motion of the tuning-fork will generally be necessary. The largest negative value of the phase-difference is $-\pi/4$ and the curve is then a parabolic arc *concave* to the fork. The significance of this is that when the fork is at its extreme outward swing, the string is also at its position of maximum displacement: a paradoxical result not in accordance with the ordinary ideas of the experiment.

A glance at the Lissajous figures for the interval of the octave pictured in fig. 7 of Lord Rayleigh's 'Theory of Sound' will show that it is possible for two similar and similarly situated curves to represent different relative phases of motion between its components, if the moving point describes the two curves in opposite directions. In order therefore to verify the phase-relation experimentally it is necessary, in addition to observing its shape, to note the direction in which the curve of motion is described. This may be done by observation through a stroboscopic disk which is kept revolving at a speed slightly less than that at which it would give one stationary view of the vibrating string. It is then fairly easy to make out the direction in which a fragment of a silvered bead attached to a point on the string near the tuning-fork describes the curve compounded of its motions longitudinal and transverse to the string. The observed direction agrees with that indicated by theory.

There is another way of writing the equations of motion which

is very useful in that it gives a clearer view of the whole case and leads us on to the subject of the next chapter. Neglecting the terms in A_3, B_3 , etc. we may put $u = P \sin (pt + E)$. Equation (5) may be written as under—

$$\begin{aligned} \ddot{u} + k\dot{u} + \left(n^2 + \frac{\beta P^2}{2}\right) u \\ = \left[2a \sin 2pt + \frac{\beta P^2}{2} \cos (2pt + 2E)\right] u \\ = aP \cos (pt - E) + \frac{\beta P^3}{4} \sin (pt + E) \end{aligned}$$

if trigonometrical functions of $3pt$ are neglected This may be succinctly written in the form—

$$\ddot{u} + k\dot{u} + N^2 u = a_1 P \cos (pt - E_1) \quad (9)$$

This is the ordinary form of the equation of a system subject to forced vibrations, and if Lord Rayleigh's equation, see (1) above, had been treated in the same way, we should have obtained

$$\ddot{u} + k\dot{u} + n^2 u = aP \cos (pt - E) \quad (10)$$

From equations (9) and (10) it is clear that a large motion might be sustained when $p = N$ or n as the case may be, and that the maintenance of vibrations by forces of double frequency is in essence only an illustration of the general principle of resonance according to which a large motion may be set up if we have equality of periods between a system and the forces acting upon it. A comparison of equations (9) and (10) shows that the introduction of the term βu^2 on the left-hand side of (5) results in a decrease in the free period of the system and also a change in the magnitude and the phase of the restoring force acting upon it. These modifications fully account for the phenomena discussed above. Equating the work done by the force represented by the right-hand side term of either of the equations (9) or (10) in any number of complete periods of the variable tension to the energy dissipated by the friction term on the left we deduce the relation

$$kp = a \cos 2E$$

which is identical with that obtained, see (4) and (8) above, from the complete analysis.

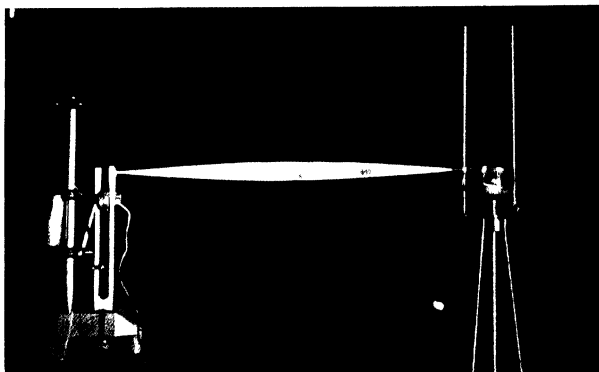


FIG. 1.

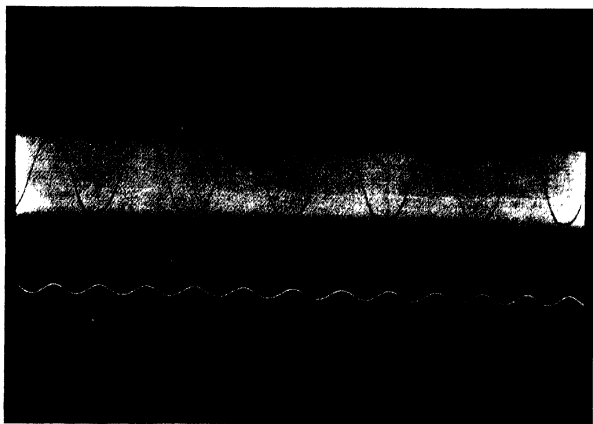
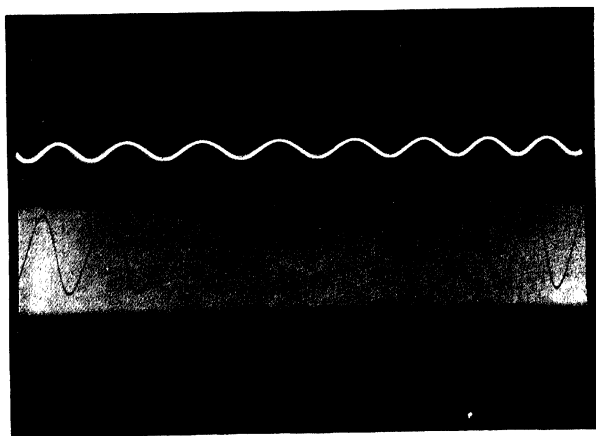
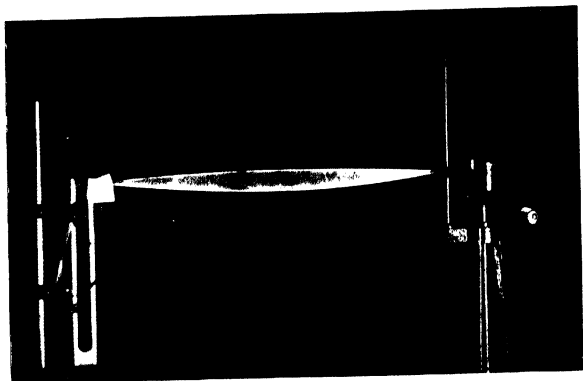


FIG. 2.

OSCILLATIONS MAINTAINED BY A VARIABLE
SPRING OF DOUBLE FREQUENCY.



MAINTENANCE OF VIBRATIONS
BY A VARIABLE SPRING OF EQUAL FREQUENCY.

IV. VIBRATION CURVES OF OSCILLATIONS MAINTAINED BY A VARIABLE SPRING.

In the last section I discussed the case of the maintenance of vibrations by forces of double frequency and emphasized the fact that in reality it only furnishes us with an illustration of the general principle of resonance according to which a periodic force acting on a system whose period is approximately equal to its own, may maintain a very considerable amplitude of motion, though in other cases its effect might be so small as to be of little account. In the course of the experimental work described in the preceding sections, I came across some other extremely interesting and remarkable cases of resonance which formed *apparent* exceptions to the above-stated law of approximate equality of periods. These cases I propose to discuss in the present paper. A preliminary note on this class of maintained vibrations was published by me in "Nature" of the 9th December, 1909, and another (illustrated) in the issue of the 10th February, 1910 (see Bulletin No. 2 of the Indian Association).

The title of this section gives an indication of the character of the forces whose action we now proceed to discuss. They only alter the 'spring' or restitutional coefficient of the system and do not tend directly to displace it from the position of equilibrium. My observations showed that there were several *quite distinct* cases in which periodic forces of this character acting on a system set up a large motion. These cases may be tabulated as follows :—

- | | | |
|-----|---|---|
| (1) | When the period of the force is $\frac{1}{2}$ | that of the system. |
| (2) | " " " " " | $\frac{2}{2}$ times that of the system. |
| (3) | " " " " " | $\frac{3}{2}$ " " " |
| (4) | " " " " " | $\frac{4}{2}$ " " " |
| (5) | " " " " " | $\frac{5}{2}$ " " " |
| (6) | " " " " " | $\frac{6}{2}$ " " " |
| (7) | " " " " " | $\frac{7}{2}$ " " " |
| &c. | &c. | &c. &c. |

Each of these forms a distinct type of maintained motion which can be obtained and studied separately by itself. The first is evidently identical with the case of 'double frequency' which was discussed in the preceding section.

To obtain any one of these types of motion, we adopt a procedure very similar to that by which the maintenance of vibrations by forces of double frequency is secured. Using a stretched string as our 'system' we subject it to a periodic variation of tension by attaching it to a tuning-fork whose prongs vibrate in a direction parallel to the string. The tension of the string (the length of which should be suitable) is adjusted so that its period of vibration in a given mode (for instance in its fundamental mode) bears the required ratio to the period of vibration of the tuning-fork. It will then generally be found that the equilibrium position of the string becomes unstable and it settles down into a state of permanent and (in suitable circumstances) vigorous vibration in which the number of swings (to and fro) made by it per unit of time bears the desired ratio to the frequency of the tuning-fork.

Each of these types of vibration presents some very remarkable peculiarities, a study of which enables us to explain the manner in which the maintenance is effected in a simple and intelligible way. When I first observed some of these types I proceeded to investigate them by precisely the same methods which I applied to the case of double frequency, i.e. mechanical or optical composition of the motion of the string with that of the tuning-fork. It is obvious that the methods are applicable to all these cases and in fact in some respects, e.g. for a detailed investigation of the phase of the maintained motion, they are probably superior to other methods of investigation that could be devised. It is evident that the Lissajous figure seen gives us at once the requisite information as regards the frequency and phase-relation between the motion of the string and that of the fork. For purposes of demonstration, however, the method of 'vibration curves' which I shall now proceed to discuss, yields far more striking and impressive results.

By the 'vibration curve' of an oscillation I mean of course its time-displacement diagram, or an equivalent thereof. To enable

the vibration curves of the oscillation of the string and that of the tuning-fork to be recorded side by side for comparison, the following arrangement was found the most suitable. Two slits were used as sources of light. One of them was horizontal and the other which was vertical was placed immediately behind the oscillating string. Both the slits were illuminated by sunlight and had collimating lenses in front of them. The fork stood with its prongs vertical and a small silvered mirror was attached with wax to the side of one of the prongs, and this of course tilted periodically through a small angle when the fork was in vibration. The light issuing from the horizontal slit was incident in a nearly normal direction upon this mirror, and after suffering reflexion at it fell upon the lens (having an aperture of $1\frac{1}{2}$ inches diameter) of a roughly constructed camera. The light issuing from the vertical slit in a direction at right angles to that from the other was deflected through 90° by reflexion at a fixed mirror and also fell upon the lens of the camera. In the focal plane of the latter was placed a metal plate with a vertical slit cut in it. The images of the horizontal and vertical slits fell one immediately above the other on the slit in the plate. Of the former only a very small length, i.e. practically only a point of light, passed through to fall upon the ground-glass of the camera or its substitute, the photographic plate. Immediately below it was the narrow image of the vertical slit crossing which was seen the shadow of the string when at rest. To photograph the vibration curves the ground-glass was removed and the dark slide which held the plate was moved as uniformly as possible by hand in horizontal grooves behind the slit in the focal plane of the camera. In the positive reproductions, the vibration curve of the string appears as a dark curve on a bright ground and that of the tuning-fork vice versa. We now proceed to consider each type of maintained oscillation and its vibration curves separately.

The first type.

Plate III shows photographs of the well-known case of the string maintained in a vibration of half the frequency of the tuning-fork, and of its vibration-curves from which it is evident at once that the tuning-fork makes two vibrations for every oscillation of

the string. The photograph shows the maximum displacements of the string to have occurred almost exactly at the epochs of minimum tension, from which we may infer, since the amplitude of motion of the string was by no means very large, that the initial tension of the string was in excess of the theoretical value, *vide* Section III of this Bulletin. We shall now consider in some detail

The second type.

The frequency of the oscillation of the string is in this case the same as that of the fork which varies its tension. This type is shown in fig. 1, Plate IV. The string vibrates in its fundamental mode, but it will be noticed that its curvature at one of the positions of maximum displacement is greater than at the other. Fig. 2, Plate IV and Plate V, show the vibration curves of this type of oscillation, and it is clear that the frequency of the motion of the string and of that of the fork are equal. In securing the photograph shown in Plate IV it was arranged that the string when at rest should exactly bisect the slit. It will be seen that its vibration curve has been displaced bodily towards one side of the slit and is thus nearer the other curve. The significance of this is that the mid-point of its oscillation is displaced to one side of the equilibrium position of the string. This is confirmed by direct observation and accounts for the greater curvature of one of the positions of maximum displacement observed in fig. 1, Plate IV. The transverse motion of *each* point on the string may therefore be represented by an expression of the form

$$u = P \sin (2\pi pt + E_2) + Q \quad (1)$$

the ratio of the coefficients P and Q being practically the same for all points on the string.

A motion of the type represented by (1) above cannot exist if the oscillations of the string were 'free' and under constant tension, inasmuch as the restoring forces at the two positions of extreme displacement would not be equal and opposite. But we are dealing here with forced oscillations under variable tension. A reference to the vibration curves will show that the maximum displacements (on either side) of points on the string occur at epochs not very far removed from those of maximum and mini-

mum tension. During one half of its oscillation the string is under a tension which is less than its normal value and during the other half under a tension which is correspondingly in excess. During the former half the motion being under diminished constraint swells out and increases in amplitude and during the other half the reverse is the case. The net result is that while the simple harmonic character of the motion is not generally departed from to any very considerable extent, the oscillation appears to take place about a point displaced to one side of the position of equilibrium, in the manner indicated by equation (1) above.

We are now in a position to understand in what manner the maintenance is effected in this case. We may write the equation of motion of a system having one degree of freedom and subject to a variable spring thus

$$\ddot{u} + k\dot{u} + n^2u = 2au \sin 2pt. \quad (2)$$

Substituting $P \sin (2pt + E_2) + Q$ for u in the right-hand side of this equation, we get

$$\begin{aligned} \ddot{u} + k\dot{u} + n^2u &= 2aP \sin 2pt \sin (2pt + E_2) + 2aQ \sin 2pt \\ &= 2aQ \sin 2pt + aP \cos E_2. \end{aligned} \quad (3)$$

if we neglect trigonometrical functions of the angle $4pt$. The first term on the right represents transverse periodic forces acting on each element of the string which would maintain a large motion having the same frequency as that of the fork if n is approximately equal to $2p$. The second term stands for a system of constant forces impressed transversely at each point on the string under the action of which the mean point of the maintained motion is displaced to one side of the equilibrium position. This is just what we get. We assumed that $u = P \sin (pt + E_2) + Q$ and the importance of the term Q is sufficiently clear from what has been said above. Substituting for u in the left side of equation (3) we get the following relations :—

$$\tan E_2 = \frac{2kp}{4p^2 - n^2}. \quad (4)$$

$$Q^2/P^2 = a^2 \cos^2 E_2/n^4 = (n^2 - 4p^2)/2n^2. \quad (5)$$

$$n^2[(n^2 - 4p^2)^2 + 4k^2p^2] = 2a^2(n^2 - 4p^2). \quad (6)$$

These equations represent the relations that must be satisfied if maintenance is to be possible. They are a fair approximation to the truth so long as the phase-difference E_2 is small. It is necessary, however, to consider the question whether the effect of terms containing trigonometrical functions of $4pt$ can be entirely ignored, particularly when according to the above formulae, Q becomes very small; which is the case when the phase-difference approaches the value $\pi/2$. We have already seen that the right-hand side of equation (3) contains such terms. We may therefore write

$$u = P \sin (2pt + E_2) + Q + R \sin (4pt + E_4) \quad (7)$$

where the ratios $P : Q : R$ are the same at all points on the string. Substituting this on the right of equation (2) we get

$$\begin{aligned} \ddot{u} + k\dot{u} + n^2u = & aP \cos E_2 + 2aQ \sin 2pt + aR \cos (2pt + E_4) \\ & - aP \cos (4pt + E_2) \end{aligned} \quad (8)$$

Each of the terms on the right of this equation represents a system of transverse forces, the effect of which we may consider separately. The first and the second we have already dealt with. The effect of the third depends upon its phase, i.e. upon the value of E_4 . This can be found by considering the action of the component of the restoring force represented by the fourth term, which has a frequency approximately double that of the free oscillation of the string. Its effect should therefore be small and should have a phase exactly opposite to that of the force producing it. By substituting for u in equation (8), we find

$$(n^2 - 16p^2)R \sin (4pt + E_4) = -aP \cos (4pt + E_2)$$

and E_4 is equal to $E_2 + \pi/2$. The third term on the right of equation (8) is therefore to $-aR \sin (2pt + E_2)$ and being exactly opposite in phase to the principal part of the motion dealt with, i.e. $P \sin (2pt + E_2)$ cannot assist in maintaining it. Its effect is merely equivalent to an alteration in the free period of oscillation of the string, and the motion is maintained entirely by the force proportional to Q represented by the second term. We have then the following relations which must be satisfied for the assumed state of steady motion to be possible.

$$\tan E_2 = -\cot E_4 = \frac{2kp}{4p^2 - N^2} \quad (9)$$

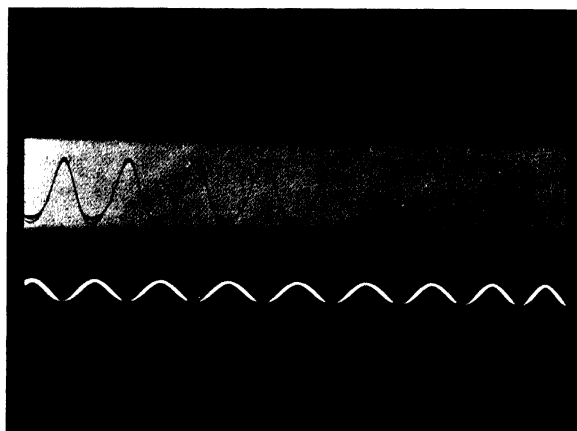
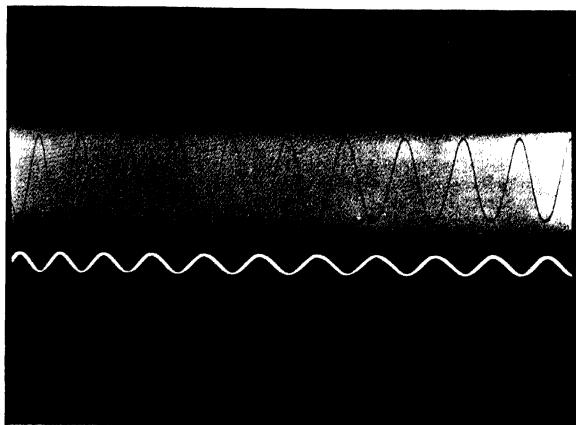


FIG. 2.

VIBRATION-CURVES OF OSCILLATIONS
MAINTAINED BY A VARIABLE SPRING OF EQUAL FREQUENCY.

$$\frac{Q^2}{P^2} = \alpha^2 \cos^2 E_2 / n^2 = (N^2 - 4p^2) / 2n^2 \quad (10)$$

$$R/P = L/12p^2 \quad (11)$$

$$n^2 [(N^2 - 4p^2)^2 + 4k^2 p^2] = 2\alpha^2 (N^2 - 4p^2) \quad (12)$$

where

$$N^2 = n^2 + L^2/12p^2$$

From formulae (10) and (11) it appears that the term $R \sin(4pt + E_2)$ in the expression for the displacement is quite appreciable (having an amplitude at least $\frac{1}{3}$ that of the term Q), though it does not assist in the maintenance of the motion. Fig. 2, Plate V, shows this component in the motion quite clearly. The vibration-curves for all points on the string exhibit the harmonic to an equal degree. According to equations (9) to (12) above, the phase of the maintained motion is independent of the amplitude, the latter quantity being indeterminate and the adjustment of pitch must be absolutely rigorous for steady vibration. All these inferences are however subject to modification in practice. The tension of the string in free oscillations of large amplitude is not constant, generally increasing by a quantity proportional to the square of the motion, and the necessary adjustment of pitch may therefore be secured by an alteration of the amplitude of motion. With however a heavy or long string horizontally held and under moderate tension, the effect of gravity is not negligible and the law of variation of the tension with the amplitude varies with the plane of the oscillation. If this is in a vertical plane, the fact that the equilibrium position is a catenary of small curvature becomes of some importance, particularly when the vibration of the string is in its fundamental mode. The tension in free oscillations of sensible amplitude would be of the form

$$n^2 + \beta(u - a)^2$$

if they occur in a vertical plane and of the form

$$n^2 + \beta u^2$$

if in a horizontal plane.

If u is put equal to $P \sin(2pt + E_2) + Q$ it is evident that it is not open to us indifferently to alter the signs of both P and Q together and retain the conditions of the motion unchanged in the former case as would be possible in the latter. The average

tension during the motion as given by the first formula would evidently be greater when Q is negative, i.e. directed downwards than when it is in the opposite direction. This appears to be the reason why as in fig. 1, Plate IV, the oscillation generally sets itself so that of the two extreme positions of the string the one which has the greater curvature is concave upwards.

Again

$$[n^2 + \beta(u - a)^2]$$

may when expanded be written in the form

$$\begin{aligned} n^2 + \beta[P^2/2 + (Q - a)^2] + \beta P[2(Q - a) \sin(2pt + E_2) \\ - P/2 \cos(4pt + 2E_2)] \end{aligned} \quad (13)$$

Of the two periodic terms the first has the same frequency as the variation of spring imposed on the system and no doubt plays an important part in the adjustment of the phase-relation between the motions of the fork and the string. In view of the fairly complete discussion of similar effects in the case of double frequency (Section III) we need not pause to consider further detail, but proceed to discuss

The third type of motion.

This is shown in fig. 1, Plate VI. Fig. 2, Plate VII, and fig. 1, Plate VIII, represent the vibration-curves of the string and the fork in cases coming under this class. The string makes three swings for every two vibrations of the fork, but the swings are not all of equal amplitude. This is evident from the vibration-curves and also from the appearance of the string itself in the first of the photographs. In addition to the two extreme positions, the photograph shows clearly two intermediate resting points of the string, one on each side of its equilibrium position, which mark the limits of the swings made at the epochs when the tension of the string is in excess. The two outer resting points, as can be seen from the vibration-curves, correspond almost exactly with the epochs of minimum tension at which the vibration being under diminished constraint swells out and increases in amplitude. The motion at any point of the string is capable of being very approximately represented by two terms. Thus—

$$u = P \sin(3pt + E_3) + Q \sin(pt + E_1) \quad (14)$$



FIG. 1.
THE THIRD TYPE OF MAINTENANCE

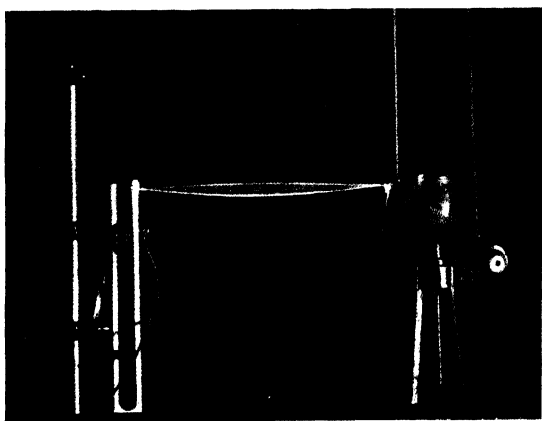
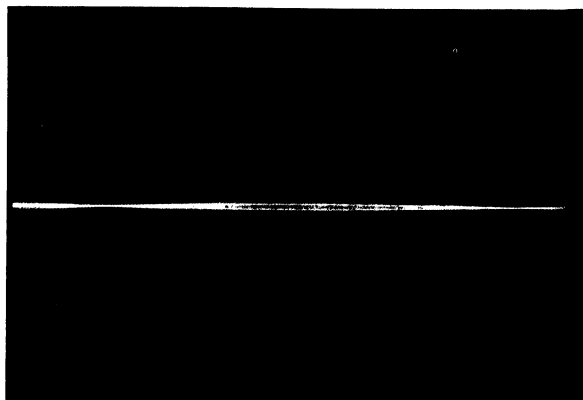


FIG. 2
THE FOURTH TYPE OF MAINTENANCE



THE FIFTH TYPE OF MAINTENANCE.

FIG. 2

VIBRATION CURVES OF THE THIRD TYPE OF MAINTENANCE.

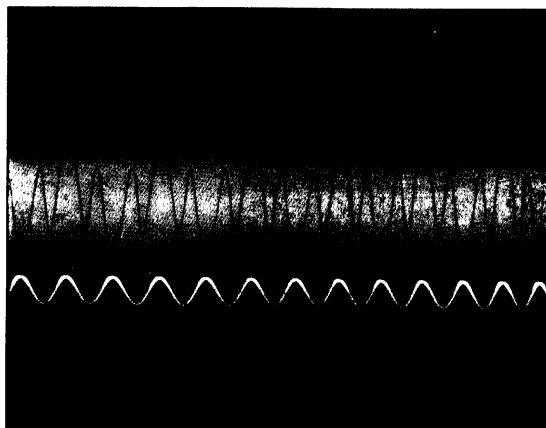


FIG. 1.
VIBRATION CURVES OF THE THIRD TYPE OF MAINTENANCE

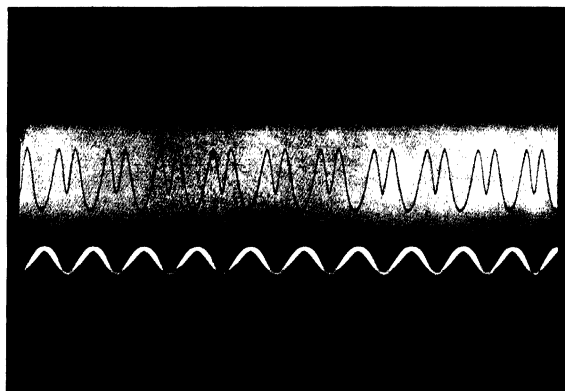


FIG. 2.
VIBRATION CURVES OF THE FOURTH TYPE OF MAINTENANCE

the second of which has the smaller amplitude and frequency and is brought into existence under the action of the variable tension. The ratio P/Q is the same at all points on the string, the motion of which may therefore be discussed as if it had only one degree of freedom. The frequency of the second term is less than that of the first by a quantity which is itself the frequency of the variable spring. The analogy between the motion as shown in the vibration-curves and that in the atmospheric 'beats' of two simple tones, one of which has the smaller amplitude and frequency, is fairly clear (see Helmholtz's 'Sensations of Tone,' Appendix XIV). The time which is required for one swing undergoes periodic fluctuations, being greatest when the tension is least and vice versa. This corresponds to the periodic flattening and sharpening of the 'beats.'

It can be shown that the maintenance of the vibrations is effected entirely by the aid of the periodic component of lower frequency, i.e. $Q \sin (pt + E_1)$, in the expression for the steady motion under variable spring. The product of this term into the variable spring gives a transverse periodic force acting on the system, which is of the right frequency and phase for maintaining its vibrations. The equation of motion may be written

$$\ddot{u} + kn + n^2 u = a [P (\cos \overline{pt + E_3} - \cos \overline{5pt + E_3}) + Q (\cos \overline{pt - E_1} - \cos \overline{3pt + E_1})] \quad (15)$$

the fourth and last term on the right representing the force referred to above. Substituting for the terms on the left and reducing, we get four equations which give us the values of E_1, E and of the ratio $\frac{Q}{P}$ and leave us in addition a relation between the 'constants' involved. The first equation is

$$3kpP = -aQ \cos (E_3 - E_1) \quad (16)$$

and this expresses the relation between the energy supplied and the energy dissipated in any number of complete periods of the variable spring. The phase-difference ($E_3 - E_1$) may be eliminated with the aid of the second relation

$$\tan (E_3 - E_1) = \frac{n^2 - 9p^2}{3kp} \quad (17)$$

In practice, as can be seen from the vibration-curves, E_3 is nearly equal to $-\pi/4$ and E_1 nearly equal to $+\pi/4$. $\cos(E_3 - E_1)$ is therefore nearly equal to zero. It is evident from (16) that $k\rho$ is very small compared with α

The third relation is

$$\tan E = \frac{(n^2 - \rho^2) - (\alpha + k\rho) \tan E_1}{(\alpha - k\rho) - (n^2 - \rho^2) \tan E_1} \quad (18)$$

This may be simplified and written as

$$\tan E_3 = (8\rho^2 - \alpha \tan E_1) / (\alpha - 8\rho^2 \tan E_1)$$

It is of interest to note that the ratio between the amplitudes P and Q is of the same order of quantities as the ratio between the constant and variable parts of the spring. It can be readily shown that to a first approximation

$$Q = \frac{9\alpha P}{8n^2 - 9\alpha} \quad (19)$$

Finally we get the relation between the constants involved by eliminating P , Q and $(E_3 - E_1)$ between the three equations (16), (17) and (19). Neglecting quantities of the order α^4 we get

$$n - 3\rho = \frac{\alpha^2}{n^2} \left(\frac{9}{16} + \frac{81}{128} \frac{\alpha}{n^2} \right) \quad (20)$$

which gives us an idea of the accuracy in adjustment of pitch that is required. In deducing this relation it is assumed that $3k\rho$ is of the order α^2/n^4 , and this is necessary if the motion is to be maintained.

It remains to consider the effect of the force represented by the term $-\alpha P \cos(5\rho t + E_3)$ on the right of equation (15). For this purpose we start afresh and assume that

$$u = A_1 \cos \rho t + A_3 \cos 3\rho t + A_5 \cos 5\rho t + B_1 \sin \rho t + B_3 \sin 3\rho t + B_5 \sin 5\rho t \quad (21)$$

Substituting in the equation of motion

$$\ddot{u} + k\dot{u} + n^2 u = 2\alpha u \sin 2\rho t$$

we get the following relations

$$A_5 = \alpha B_3 / 16\rho^2, \quad B_5 = -\alpha A_3 / 16\rho^2 \quad (22)$$

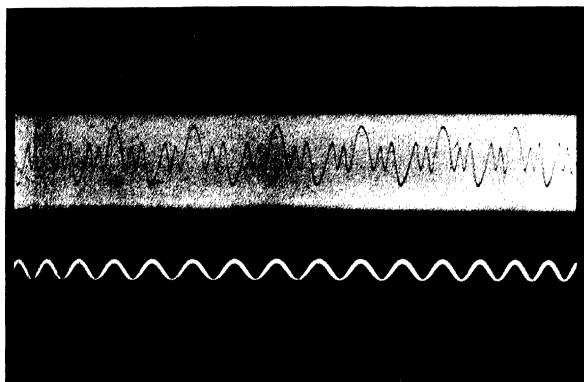


FIG. 1.
VIBRATION CURVES OF THE FIFTH TYPE OF MAINTENANCE

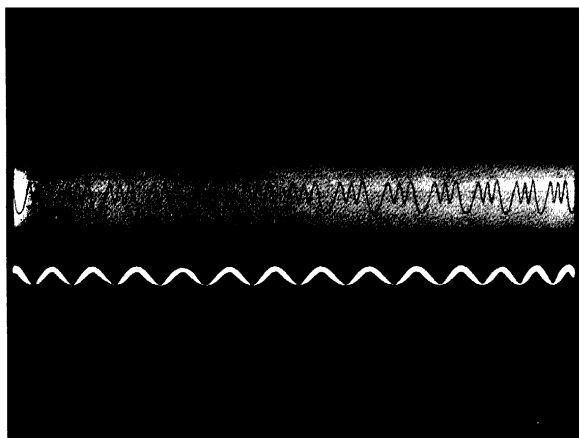


FIG. 2.
VIBRATION CURVES OF THE SIXTH TYPE OF MAINTENANCE

$$\left. \begin{aligned} 3kpA_3 - (n^2 - 9p^2 + a^2/16p^2)B_3 &= -aA_1 \\ (n^2 - 9p^2 + a^2/16p^2)A_2 + 3kpB_3 &= -aB_1 \end{aligned} \right\} \quad (23)$$

$$\left. \begin{aligned} (a + kp)A_1 - (n^2 - p^2)B_1 &= aA_3 \\ (n^2 - p^2)A_1 - (a - kp)B_1 &= aB_3 \end{aligned} \right\} \quad (24)$$

Equations (18) and (24) are identical and by comparing equations (17) and (23) it can readily be seen that the only change is that instead of n^2 we have the very slightly larger quantity $(n^2 + a^2/16p^2)$ in the latter equation. The inference is that the component in the displacement which has a frequency higher than that of the principal part of the motion does not assist in its maintenance, the effect produced by it being merely equivalent to a very small decrease in the free period of the oscillation of the string. It will be recollected that a similar result was obtained in the case of the second type of the maintenance of vibrations discussed above. From equation (22) it appears that the amplitude of the component of frequency $5p/2\pi$ in the maintained motion is less than half the amplitude of the component of frequency $p/2\pi$. If the former is represented by $R \sin(5pt + E_5)$ it is evident from what was said above that E_5 is nearly equal to $\pi/4$. We have roughly

$$u = P \sin(3pt - \pi/4) + Q \sin(pt + \pi/4) + R \sin(5pt + \pi/4)$$

when $pt = \pi/4$, $u = (P + Q - R)$, and when $pt = 5\pi/4$, $u = -(P + Q - R)$.

The maximum amplitudes are therefore less than they would be if the component R did not exist. Its presence should therefore render itself evident by a flattening of the vibration-curve at the epochs of minimum tension. Some flattening of this kind, though not very marked, appears to be shown in fig. 2, Plate VII.

The preceding discussion gives us only the phases and the ratios of the amplitudes of the components of the maintained motion, and according to the equations the actual amplitudes are indeterminate. In practice however the equation of motion is subject to modification on account of the variation of tension in free oscillations of sensible amplitude. For simplicity we may consider a case in which the string is vertical. The effect of gravity on its transverse oscillations may then be neglected, and the equations of motion may be written in the form

$$\ddot{u} + k\dot{u} + (n^2 + \beta u^2 - 2a \sin 2pt)u = 0$$

If $u = P \sin (3pt + E_3) + Q \sin (pt + E_1) + R \sin (5pt + E_5), n^2 + \beta u^2$
 $= n^2 + \beta(P^2 + Q^2 + R^2)/2 + \text{a large number of periodic terms.} \quad (25)$

As may have been expected, the average tension is increased by a large amplitude of motion and this is no doubt what secures the necessary adjustment of pitch and determines the amplitude of the maintained vibration. Of the periodic terms in (25) probably the most important are those which have a frequency equal to that of the imposed variable spring and tend directly to alter its magnitude or effectiveness. There are only three such terms, and if we neglect the others,

$$n^2 + \beta u^2 = n^2 + \beta(P^2 + Q^2 + R^2)/2 + \beta QP \cos 2pt + E_3 - E_1 \\ - \beta Q^2 \cos 2pt + 2E_1 + \beta RP \cos 2pt + E_5 - E_3.$$

Putting $E_3 = -\pi/4$ and $E_1 = E_5 = \pi/4$ approximately, we have as our equation of motion

$$\ddot{u} + k\dot{u} + [N^2 - (2a - Q(P + Q) + RP) \sin 2pt] u = 0.$$

From this it seems evident that when the imposed variation of tension is in excess of that just required to maintain the motion, the component of amplitude Q tends to increase at the expense of the component with amplitude R . The latter tends to become even less important than it would otherwise be, and indeed there does not appear to be any very marked indication of its existence in the vibration curves.

The fourth type of motion.

This is shown as fig. 2, Plate VI, and its vibration curve as fig. 2, Plate VIII. In the former a resting point intermediate between the two extreme positions of the string is clearly visible, and it can be seen in the vibration curve that this corresponds exactly to the terminal point of the swing at the epoch of maximum tension. The string makes four swings for every two vibrations of the fork. Of these the two occurring when the tension is in excess are less in amplitude and take a shorter time than the two others made when the tension is in defect. The maintained motion consists therefore of a principal component of frequency double that of the fork, the simple harmonic character

of which is modified under the action of the variable spring and which therefore appears along with a subsidiary motion of the same frequency as that of the fork. As in the previous cases discussed, the ratio between the two components is practically the same at all points of the string, and the problem may therefore be dealt with as if it related to a system with one degree of freedom only.

For a full discussion we must assume that the displacement at any instant may be represented by an expression of the form

$$u = A_2 \sin 2pt + A_4 \sin 4pt + A_6 \sin 6pt + B_0 + B_2 \cos 2pt + B_4 \cos 4pt + B_6 \cos 6pt \quad (26)$$

The terms of frequency $4p/2\pi$ form the principal part of the maintained motion, and these and the terms of lower frequency $2p/2\pi$ are, as can be seen from the vibration curve, predominant. As in the previous cases discussed, it can be shown that the latter terms are mainly instrumental in maintaining the motion, in other words that their product into the variable spring gives a transverse periodic force of the right frequency and phase for maintaining the vibrations. The components of frequency $6p/2\pi$ have an influence on the motion of the system which is equivalent merely to a slight decrease in the period of the free vibrations of the string, and they do not otherwise assist in the maintenance of the vibrations. The constant term B_0 though small is by no means negligible, and it remains to investigate its influence in the present case. As shown in the investigation of the second type of motion, a term of this kind may be regarded as the result of a system of constant forces acting at all points on the string. To solve the equation of motion, we substitute (26) for u in the formula—

$$\ddot{u} + k\dot{u} + n^2u = 2au \sin 2pt$$

The following relations are obtained—

$$A_6 = -aB_4/20p^2, \quad B_6 = aA_4/20p^2 \quad (27)$$

$$\left. \begin{aligned} (n^2 - 16p^2 + a^2/20p^2)A_4 - 4kpB_4 &= aB_2 \\ 4kpA_4 + (n^2 - 16p^2 + a^2/20p^2)B_4 &= -aA_2 \end{aligned} \right\} \quad (28)$$

$$n^2B_0 = aA_2 \quad (29)$$

$$\left. \begin{aligned} (n^2 - 4p^2)A_2 - 2kpB_2 &= a(2B_0 - B_4) \\ 2kpA_2 + (n^2 - 4p^2)B_2 &= aA_4 \end{aligned} \right\} \quad (30)$$

It is not permissible to leave out B_0 in the first of the equations (30), for, if we do so and eliminate A_2 , B_2 , A_4 and B_4 between the equations (28) and (30) we get an eliminant of the form $S^2 = -T^2$ which is evidently absurd. The significance of this may be understood by writing equations (28) in the form—

$$4kp(A_4 + B_4)^{\frac{1}{2}} = -a(A_2 + B_2)^{\frac{1}{2}} \cos(E_2 - E_4) \quad (31)$$

This formula expresses the relation that the energy dissipated by friction in a time comprising any number of complete periods of the variable spring is equal to that supplied in the same time through its agency. Now it can be seen from the vibration curve that $(E_2 - E_4)$ is very nearly equal to $\pi/2$ and $\cos(E_2 - E_4)$ is therefore very small, but still sufficient to sustain the motion. If in the first of the equations (30) we neglect B_0 , the value of $(A_2 + B_2)^{\frac{1}{2}}$ and $(A_4 + B_4)^{\frac{1}{2}}$ is not very appreciably affected, but $\cos(E_2 - E_4)$ is however reduced to such an extent that it is no longer possible for equation (31) to be satisfied, in other words the motion cannot be maintained. The term B_0 is equal to aA_2/n^2 , i.e. equal to $-4a^2B_4/3n^4$ and is therefore very small, but as explained above, the maintenance of the vibrations cannot be fully explained without taking it into account.

The components A_4 and B_4 in the maintained motion are both very small. The component A_2 is approximately equal to $-4aB_4/3n^2$. The ratio between the principal part of the maintained motion and the subsidiary component of lower frequency is therefore of the same order of quantities as the ratio between the constant and variable parts of the spring.

The fifth type of motion and the general case.

Fig. 1, Plate VII, and fig. 1, Plate IX, show this class of maintained motion and its vibration curves respectively. In this case, the frequency of the variable spring is two-fifths that of the free oscillations of the system. The forced oscillations of the system may be discussed as if it possessed one degree of freedom only, the displacement at any point on the string being given by an expression of the form

$$u = A_1 \sin pt + A_3 \sin 3pt + A_5 \sin 5pt + A_7 \sin 7pt + B_1 \cos pt + B_3 \cos 3pt + B_5 \cos 5pt + B_7 \cos 7pt \quad (32)$$

the variable spring being as in previous cases represented by $-2a \sin 2pt$, and the ratio of the constants being the same for all points on the string. The principal part of the maintained motion is $(A_5 + B_5)^{\frac{1}{2}} \sin(5pt + E_5)$, which is very approximately of the same frequency as the free oscillations of the system. It can be seen from the vibration curve that E_5 is approximately equal to $-\frac{3\pi}{4}$ and A_5 is therefore nearly equal to B_5 .

The subsidiary term $(A_3 + B_3) \sin(3pt + E_3)$ in the motion is from a physical point of view of great importance. It is not at all difficult to understand in what manner it is brought into existence. The successive oscillations of the string are evidently not all executed under identical conditions. At the epoch of minimum tension the motion being under diminished constraint swells out and increases in amplitude and the contrary is the case at the epochs of maximum tension. Again at the former epochs the time taken for a swing is more than at the latter. The motion as shown in the vibration curves is very analogous to the effect of 'beats.' Taking the general case in which a variable spring $-2a \sin 2pt$ acts upon a system whose free oscillations have a frequency nearly equal to $r/2$ times that of the variable spring, the frequency of the 'beats' is equal to that of the variable spring and the frequency of the subsidiary motion is less by that quantity than the frequency of the principal motion, and we may therefore put

$$W = P \sin(rpt + E_r) + Q \sin(r - 2pt + E_{r-2}) \quad (33)$$

The product of the variable spring with the displacement at any instant may be regarded as the impressed part of the restoring force. Taking the first term on the right of (33), the product $-2aP \sin 2pt \sin(rpt + E_r)$ has no component of the frequency $rp \mid 2\pi$ which is required (by the general principle of resonance) if the oscillation is to be maintained. On the other hand the product

$$-2aQ \sin 2pt \sin(r - 2pt + E_{r-2})$$

does contain such a component which is equal to

$$aQ \cos(rp + E_{r-2})$$

and can maintain the motion if the other conditions are suitable.

The energy dissipated in any number of complete periods of the variable spring is equal to the energy supplied during the same interval if

$$r k p P = -a Q \cos (E_{r-2} - E_r). \quad (34)$$

This equation conveys the fundamental principle underlying the type of the maintenance of vibrations under discussion. In the general case Q is of the order

$$\pm \frac{aP}{4(r-1)p^2},$$

if we neglect possible effects due to the variability of the tension in free oscillations of sensible amplitudes. In order to show that the value of $\cos (E_{r-2} - E_r)$ in equation (34) may be sufficiently large to ensure maintenance of the motion, it is necessary to consider the effects produced by terms of still smaller frequencies (if any) in the expression for the displacement at any instant. An illustration of this point has already been given in the case of the fourth type of motion. Such terms exist in all cases whose $r > 3$. They owe their origin to secondary and tertiary reaction between the forced oscillations and the variable spring, and though very small in magnitude play an important part in building up the requisite phase-difference between the principal motion and its immediate auxiliary of lower frequency. Thus, returning to the case of the fifth type discussed above, we cannot neglect the term

$$(A_1^2 + B_1^2) \sin (pt + E_1)$$

in the expression for the displacement, for if we do we should find that the value of $\cos (E_3 - E_6)$ is not sufficient to maintain the motion. On the other hand the terms

$$(A_7 + B_7)^{\frac{1}{2}} \sin (pt + E_7)$$

etc. do not play any such part in the maintenance. Their effect is merely equivalent to a slight alteration in the free period of oscillation of the string, and they are generally inconspicuous. It is hardly necessary for me to add that in each case the necessary adjustment of pitch is secured by the variation of the period of the motion with increasing amplitudes.

I have also observed the sixth and seventh and higher

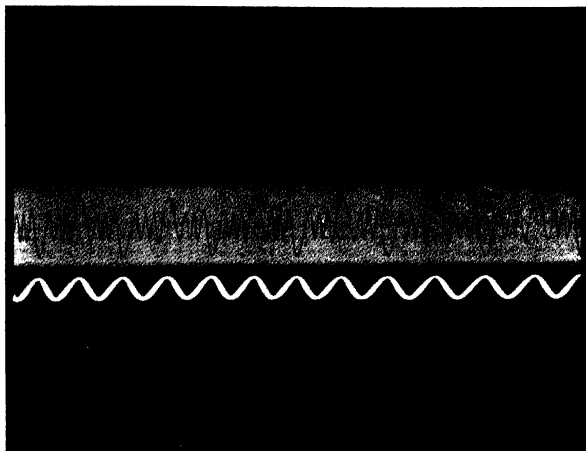


FIG. 1.
VIBRATION CURVES OF THE SEVENTH TYPE OF MAINTENANCE

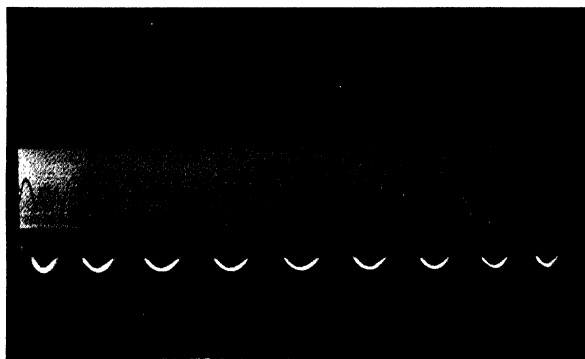


FIG. 2.
VIBRATION CURVES OF THE EIGHTH TYPE OF MAINTENANCE

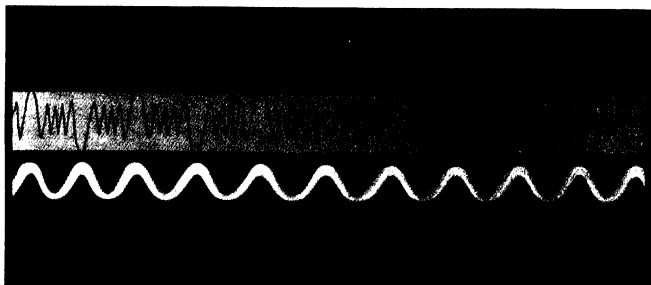


FIG. 1.
VIBRATION CURVES OF THE NINTH TYPE OF MAINTENANCE

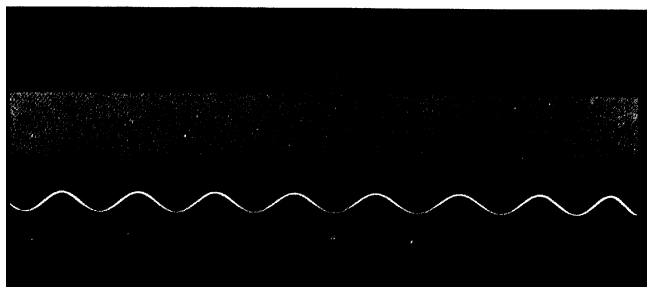


FIG. 2.
VIBRATION CURVES OF THE TENTH TYPE OF MAINTENANCE

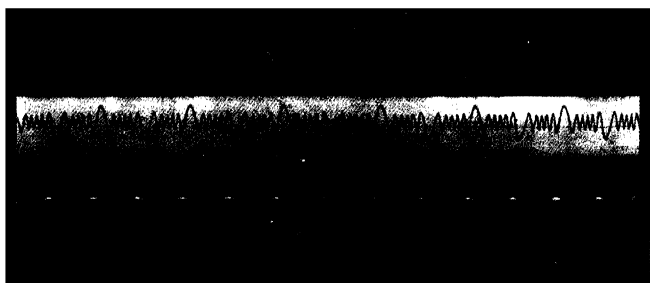


FIG. 3.

types of motion in the series up to the eleventh and have photographed their vibration-curves. These are shown as fig. 2, Plate IX and Plates X and XI. The appearance of a string executing the sixth or eighth or tenth type of motion is somewhat analogous to that in the case of the fourth type, and that of the seventh or ninth or eleventh to the fifth type. The reason for this can be well understood. For the odd types are all more or less perfectly *symmetrical* and the even types are all *unsymmetrical*.

• *Observations with Revolving Mirror and with Stroboscopes.*

That the string when maintained in any one of these types of vibration behaves as a single unit, in other words like a system having only one degree of freedom, can well be shown by observing the vibration curves at different points on the string. By illuminating any one point by a sheet of light transverse to the string and viewing the luminous line of light in a mirror kept revolving at a moderate speed, the vibration curve is seen at once. Even a mirror held in the hand which is tilted to and fro is sufficient for the purpose. Shifting the sheet of light so that it cuts the string at any other point produces no effect except to alter all the ordinates of the vibration curves in equal ratios.

Some extremely interesting phenomena are noticed when a stroboscopic disc is used in observing these types of maintained motion. A Rayleigh synchronous motor on which is mounted a blackened disc with narrow radial slits cut in it is very suitable for this purpose. As already mentioned in Section I, one of the discs which I use has thirty slits on it, the armature-wheel of the motor having the same number of teeth. The electric current from the self-interrupter fork which maintains the string in vibration also runs the synchronous motor. In making the observations, the stroboscopic disc is held vertically and the string which is set horizontal and parallel to the disc is viewed through the top row of slits, i.e. those which are vertical and move in a direction parallel to the string as the disc revolves. It is advantageous to have the whole length of the string brilliantly illuminated and to let as little stray light as possible fall upon the reverse of the disc at some distance from which the observer takes his stand. A

brilliant view is then obtained. I have already explained that under these circumstances we see the string in successive cycles of phase along its length, and the peculiar character of the maintained motion in these cases is brought out in an extremely remarkable way. *The string is seen in the form of a vibration curve*, which would be identical with those shown above, but for the fact that the amplitude of motion is not the same at all points of the string, being a maximum at the ventral segments and zero at the nodes.

Another point calls for remark. Using a fork with a frequency of 60 per second, the *free* oscillations of the string have a frequency of 30 in the case of the 1st type, 60 in the case of the 2nd, 90 with the 3rd, 120 with the 4th, 150 with the 5th, and so on. With the disc having 30 slits on it we get 60 views per second of any one point on the string, and with the even types of motion, i.e. the 2nd, 4th, etc. the 'vibration-curve' seen through the stroboscopic disc appears single. With the odd types, i.e. the 1st, 3rd, 5th, etc., *two* vibration-curves are seen, one of which is as nearly as can be seen the mirror-image of the other, intersecting it at points which lie or should lie upon the equilibrium position of the string. The reason why with the odd types we see the vibration-curve double is obvious enough, and I need not proceed to detail it. The double pattern brings home to the eye in an extremely vivid and convincing manner the fact that under the action of the variable spring the 'amplitude' and 'period' of the motion periodically increase and decrease after the manner of 'beats.'

An interesting variation on the experiment is made by using the disc with 60 slits. We then get 120 views per second and with the even types we get the vibration-curves double, but one of the curves is not the mirror image of the other, the motion not being symmetrical. On the other hand, with the odd types we see the vibration-curves in quadruple pattern, the third and fifth types in particular giving extremely beautiful tracery effects. It seems somewhat difficult to obtain perfectly satisfactory photographs of these phenomena on account of slight periodic alterations in the speed of the stroboscopic disc, but I am still quite hopeful.

V. THE MAINTENANCE OF COMPOUND VIBRATIONS BY A SIMPLE HARMONIC FORCE.

In this and the succeeding section on 'Transitional Modes of Motion under Variable Spring' I shall consider the phenomena of the maintenance of vibrations by a variable spring of simple harmonic character acting on a system that has more than one degree of freedom. In Section IV, I have shown that a variable spring acting on a system having only one normal mode of oscillation may maintain its vibrations if the frequency of the variable spring stands to that of the system in any one of the ratios $2:r$ where r is an integer. We know that a vibrating system of the kind here dealt with, i.e. a stretched string, has not merely one free period of oscillation, but a series of such free periods in which it divides up into one, two or more segments. Since the frequencies of oscillation which a variable spring of given frequency may maintain under suitable circumstances also form a series, it is evidently possible for more than one mode of vibration to be maintained at one and the same time, *each with its own appropriate frequency*. In other words, the variable spring may maintain a compound vibration, and as the components of this motion need not both or all be in one and the same plane of vibration of the string, we may readily obtain by a little calculation and trial, types of maintained motion in which the oscillation in one principal plane is of one frequency and in the perpendicular plane of a different frequency. Under these circumstances, the motion of a point on the string in a plane transverse to it becomes and remains the appropriate Lissajous figure, and the frequency relation between the component motions is thus rendered evident to inspection in a most striking manner.

The photographs shown in Plate XII represent short sections of the string thus maintained in stationary vibration, one point in the middle of the section being brilliantly illuminated. Fig. 1 shows the ordinary first type of maintenance in which the frequency of the motion is half that of the fork. Fig. 2 shows a compound of the first and second types in suitable phase relation, the motion being in a parabolic arc. Fig. 3 is a compound of the first and third types. Fig. 4 is a compound of the second and third types of frequencies respectively equal to and half as much again as that

of the fork. Figs. 5 and 6 are complementary, i.e. the same mode of vibration, fig. 5 showing one part of the string and fig. 6 another. In these two photographs the first and third types occur in one principal plane and the second type by itself in the perpendicular plane. In fig. 5, the first and third are in similar phases, but in fig. 6 they are opposed, hence the very remarkable split ring effect in the latter. In fig. 7 we have the first and third types again in perpendicular planes but along with the third type there is a clear addition of the second type as well. Figs. 8 and 9 are complementary, and show the first type maintained in one plane and the second and fourth types together in the perpendicular plane. Fig. 10 represents a compound of the second and fifth types, and shows quite clearly the characteristics of the fifth type as described in the previous section, i.e. the increase of the amplitude and period of the motion at the epoch of minimum tension and their decrease when the tension is a maximum. Fig. 11 shows the first type in one plane and the second and fifth types together in a perpendicular plane. Figs. 12 and 13 are complementary, i.e. show different parts of the string in the same mode of oscillation. They represent the first and fifth types together in one plane and the second by itself in the perpendicular plane. In fig. 12, the first and fifth types are in the same phase and in fig. 13 they are opposed. Figs. 14 and 15 show the first type in one plane and the second and sixth types together in the perpendicular plane. The two latter are in different relative phases in the two photographs.

Besides the above, I have observed a very large number of permanently maintained compound modes of vibration in which two or more of the types of motion discussed in the preceding section occur in various phase-relations to each other. In the case of types of higher order than the second, the observed range of variation of phase was not however very large.

The compound modes of motion in which two or more of the types of maintained motion occur together in one plane of vibration can also be observed stroboscopically in the manner described in the preceding section. The special feature of interest in this case is that a large number of variations can be obtained and different parts of the string, sometimes even contiguous ones, show

PLATE XII.

FIG. 4.



FIG. 5.

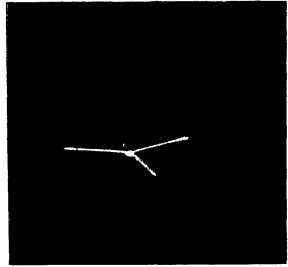


FIG. 9.

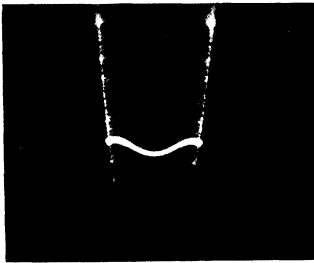


FIG. 10.

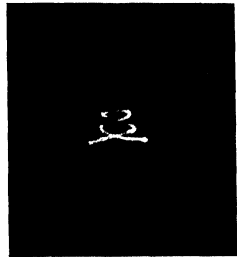


FIG. 14.

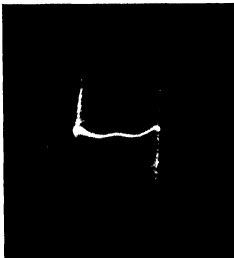


FIG. 15.



FIG. 1.

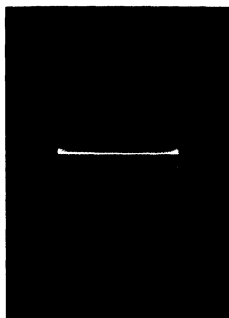


FIG. 2.

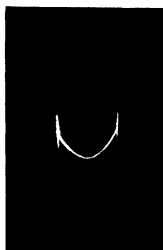


FIG. 3.



FIG. 4.



FIG. 5.

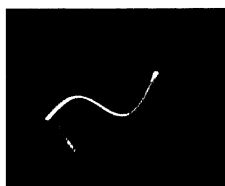


FIG. 6.



FIG. 11.



FIG. 12.



FIG. 13.



COMPOUND TYPES OF VIBRATION OF A STRETCHED STRING
MAINTAINED BY A SIMPLE HARMONIC VARIATION OF TENSION.

the component motions in different relative magnitudes and as seen through the stroboscopic disc in different phases. The double patterns obtained in this manner are extremely interesting and beautiful, and it is with regret that I decided not to delay the issue of this Bulletin till I had secured satisfactory photographs of some of them.

VI. TRANSITIONAL MODES OF MOTION UNDER VARIABLE SPRING.

In discussing the maintenance of vibrations by a variable spring of double frequency, *vide* Section III above, it was tacitly assumed that the motion was that of a system having one degree of freedom only, or at any rate could be treated as such to a close approximation. In other words it was taken for granted that the ordinary 'modes' of vibration under constant spring, i.e. the ratios of the displacements at any instant of different points on the system, remain unaltered. We are justified in making this assumption so long as the free periods of the system in its several normal modes of oscillation are sufficiently removed from each other. But as will be seen from what follows, it breaks down entirely when the frequencies of two natural modes of vibration between which the half frequency of the imposed variable spring lies are sufficiently close together to fall simultaneously within the range of maintenance for the given frequency. The phenomena that then result are of considerable experimental and theoretical interest and I have termed them 'Transitional Modes of Motion.' The appropriateness of this will appear as we proceed.

A variable spring of given frequency can maintain the vibrations of a system whose free period for small oscillations lies anywhere within a certain range determined by the magnitude of the imposed variation. If two of the normal modes of vibration of the system fall simultaneously within this range the steady motion, if any, that may result must evidently be of a frequency exactly half that of the variable spring imposed. It is possible to obtain the over-lapping of the ranges for two contiguous modes even if the free periods of these differ considerably, by sufficiently increasing the magnitude of the variable spring. It is experimentally observed that the maintenance of a steady state of vibration is perfectly

possible under such circumstances. Thus for instance, there is absolutely no difficulty in obtaining a steady transitional mode of motion which is intermediate between the ordinary modes in which a string divides up with three and four ventral segments respectively. In the resulting vibration there is nothing that can even approximately be regarded as a 'node.' The amplitude of vibration is not however the same at all points of the string, and there are recognizable maxima and minima. It is not however easy to describe the appearance seen with much exactness and to a cursory examination the nature of the motion is by no means evident. When however we use intermittent illumination of frequency nearly equal to that fork maintaining the string in vibration, the extremely remarkable and interesting character of the motion is at once revealed. Since the frequency of the illumination is nearly but not quite double that of the motion we see simultaneously two opposite phases of the motion which undergo periodic cycles of change. At one instant the string is seen in the form of recognizably perfect sine-curves which enclose *three* ventral segments. At another phase of the motion it is seen in the form of perfect sine-curves which enclose *four* ventral segments. The periodic change from three to four segments and back again is one of the most striking and interesting phenomena that are met with in the study of the maintenance of vibrations.

When the frequency of the intermittent illumination is somewhat less than that of the fork we see the motion proceeding in the manner in which it actually takes place. One way of describing what is observed is to say that extra loops are continually being formed at the end of the string attached to the fork and continually moving off and disappearing at the fixed end. The process at one end is periodically faster and slower than that at the other, with the result that we have alternately three and four ventral segments on the visible portion of the string.

There is however another fact which is observed, i.e. the ordinates of the three-loop curve are not equal to those of the four loops, being generally larger: this is not brought out in the description given above. Perhaps a more accurate idea may be conveyed in the following manner. If we have a pair of curves whose initial positions are given by the equations

$$y = \pm P \sin \frac{3\pi x}{b} \text{ and } z = \pm Q \sin \frac{4\pi x}{b}$$

and which continually rotate round the axis of x , the plane yz being normal to the latter, their motion as seen projected on any given plane passing through the axis of x is similar to that seen in the actual experiment with the intermittent illumination. The projected curves would be given by the equation

$$\pm u = A \sin \frac{3\pi x}{b} \sin (pt + \pi/4) + B \sin \frac{4\pi x}{b} \sin (pt + \theta). \quad (1)$$

If the axes of y and z are at right angles, $\theta = -\pi/4$, and the phase difference between the two terms would be exactly quarter of an oscillation.

From equation (1) it is clear that the phase of the resultant motion varies from point to point on the string. Working by the methods described in Section III, I have observed the variation of the phase of the motion along the string and the indications of equation (1) of the present section are amply confirmed. Since A is generally larger than B , the most remarkable changes are observed on either side of the points where $x=b/3$ or $2b/3$. At some distance from these points the 'curves of motion' (*vide* Section III) are parabolic arcs *convex* to the fork. As we approach nearer they become first looped figures convex to the fork and then 8-shaped curves. Nearer still, they are looped figures *concave* to the fork and finally parabolic arcs with their curvature directed towards the fork. As we recede on the other side we get the same changes in reverse order, the curves at some distance off being parabolic arcs convex to the fork. I hope later to obtain and publish photographs of these remarkable types of motion with the varying phase.

It is not difficult to see why the displacement at any point of the string is of the type given by equation (1). As already explained in the third section of this Bulletin, the maximum positive and negative phase-differences between the variable spring of double frequency and the motion maintained by it are $\pi/4$ and $-\pi/4$ respectively. When the half-frequency of the variable spring is intermediate between the frequencies of free oscillations of the system in any two given modes, we may assume that the oscilla-

tions are set up and maintained *simultaneously* in the two different modes but with the same frequency, i.e. half that of the variable spring. The two modes of oscillations are however in different phases, and the sustained vibration can well be termed a transitional mode of motion.

In concluding this Bulletin I have real pleasure in acknowledging my indebtedness to Dr. Amrita Lal Sircar for his interest in the work and unfailing personal encouragement to myself, and also for his having as Hon. Secretary put the resources of the laboratory of the Association and the services of the staff unreservedly at my disposal during hours at which few institutions, if any, would remain open for work. I have also specially to mention the name of the senior demonstrator Mr. Dey for having materially assisted in the early and rapid completion of the experimental work.

APPENDIX.

Note on a Stereo-Optical Illusion.

Working with a stroboscopic disc of the pattern already described, I noticed a very curious optical illusion that seems worthy of record. The disc with 30 slits was set up vertically, and the turning-fork which regulated its motion was placed immediately behind with its prongs vertical and facing the disc so that the observer who took his stand immediately in front of it could get a good view of the motion of both the prongs. Using both eyes for comfort, I was surprised to notice that the prongs appeared bent out of their plane, one to the front and one to the rear, and actually executed oscillations to the rear and to the front as the head was moved along the row of slits! The explanation of the phenomenon was undoubtedly that the two eyes perceived the motion of each of the prongs of the fork in two distinct phases and endeavoured to reconcile them by seeing them bent out of their plane one to the front and the other to the rear! The appearance was most realistic.

I. On the Maintenance of Combinational Vibrations by Two Simple Harmonic Forces.

By C. V. Raman, M.A.

(Plates I to VI.)

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General Discussion of Results	11
Theory of Combinational Maintenance	15

INTRODUCTORY.

In a previous publication (Bulletin No. 6 of the Indian Association for the Cultivation of Science) I dealt with my experimental investigations on a new class of forced oscillations maintained by periodic variation of spring, of which the well-known phenomenon of the maintenance of vibrations by forces of double frequency (dealt with by Faraday, Melde and Lord Rayleigh) is a specific case. Working with the same apparatus as is used for one of the forms of Melde's experiment, I showed that a simple harmonic force acting longitudinally upon a stretched string could maintain its vibrations when the frequency of the free oscillations of the string in any given mode, is sufficiently nearly equal to any integral multiple of half the frequency of the fork. To illustrate and explain the manner in which such maintenance is effected, a series of photographs were published with the paper, showing simultaneous vibration-curves of the exciting tuning-fork and the

maintained motion of the string, in which the natural frequencies of the latter were various multiples (up to the eleventh), of half the frequency of the former. From these photographs and the mathematical discussion, it became clear that a very important part in the maintenance of the motion is played by certain subsidiary components introduced into it under the action of the variable spring. The principal component of the maintained vibration together with the subsidiary motions thus introduced could, it was shown, be arranged in the form of a Fourier series, the difference of frequency between the successive terms being that of the variable spring itself.

The successful investigation of this class of resonance-vibrations suggested further experiments with systems subjected simultaneously to *two* simple harmonic forces of differing frequencies varying the spring. These have been productive of some extremely interesting results and will form the subject of the present paper.

EXPERIMENTAL ARRANGEMENTS.

As mentioned above, the idea underlying the investigation now to be described was to subject a system, having a frequency of vibration that could be adjusted to any desired value over a wide range, e.g., a stretched string, simultaneously to two simple harmonic forces of known frequencies varying its spring, and then to observe and record the various cases in which the state of equilibrium which usually obtains becomes unstable and the system settles down into vigorous vibration.

The experimental method adopted was extremely simple. Two electrically-maintained turning-forks were used. These stood on the table at some distance apart with their prongs vertical, in one plane, and directions of vibration parallel. A fine silk or cotton string, one or two metres in length, was stretched horizontally between the two forks, its extremities being attached to one prong of each fork (i.e., to those nearest to each other). The tension of the string when the forks were at rest could be readily adjusted by merely sliding one fork slightly towards or away from the other along the table. Since the prongs of the fork are vertical and the string is parallel to their direction of vibration,

we have as the result when the forks are excited, that the tension of the string is periodically varied by the vibrations of both simultaneously.

Of course, with the arrangements described neither of the forks, whether acting by itself or conjointly with the other, tends *directly* to displace the string from its position of equilibrium. They vary the tension of the string, but the latter remains undisturbed so far as transverse movement is concerned, except when the initial tension, i.e., the frequency of free oscillation of the string, is adjusted so as to coincide more or less accurately with certain values which we may for convenience term 'resonance frequencies,' leaving the justification of this phraseology to be dealt with later.

Certain of the resonance-frequencies should obviously be multiples of half the frequency of one or the other of the forks by itself. For, each of the forks acting alone can maintain a vigorous vibration in a number of cases as shown in the paper referred to above (Bulletin No. 6), and this vibration is excited and maintained under suitable conditions even in the presence of the other periodic force varying the spring. To put it mathematically, if the frequency of free oscillation of the string in any given mode is sufficiently nearly equal to either $\frac{1}{2} r N_1$ or $\frac{1}{2} s N_2$, where N_1, N_2 are the frequencies of the forks, r, s being any positive integers, we would get resonance-vibrations as already shown. That certain of the resonances observed are of this class, can readily be verified by stopping the fork which does not play a part in the maintenance, when the vibration of the other fork continues to sustain the motion of the string.

Besides the resonances of the kind described in the preceding para., the observer is surprised and delighted to find, even at a first trial of the experiment, a large number of other cases of vigorous maintenance which have evidently to be ascribed to the joint action of the two forks on the string. Their variety and number is extraordinary, and these, together with the way in which they come rapidly following one another particularly at the higher frequencies, remind the observer, by a vivid analogy, of the lines in a complicated spectrum-series. It is readily guessed at once that these are cases of 'Combinational' resonance in which the

frequency of the principal term in the maintained motion is related *jointly* to the frequencies of both the forks. This fact is readily verified by experimental investigation as described below, and the results obtained can be stated with generality thus. Under suitable conditions the equilibrium of the system becomes unstable and a vigorous motion is maintained if the frequency of free vibration in any given mode is sufficiently nearly equal to $\frac{1}{2} r N_1 \pm \frac{1}{2} s N_2$, where r and s are *positive* integers. The degree of accuracy of adjustment necessary for maintenance increases as r and s increase. Where the positive sign applies we have 'summational' resonances. With the negative sign we have 'differential' resonances. The frequency of the maintained motion is *exactly* equal to $\frac{1}{2} r N_1 \pm \frac{1}{2} s N_2$, where r and s have the values assigned.

Of course N_1 and N_2 , which are the frequencies of the forks, do not in general stand in any simple arithmetical ratio, and the cases of 'Combinational' resonance described in this paper could in almost all cases be recognized and distinguished from 'simple' resonance due to either of the forks acting alone by a peculiar appearance of 'flicker' due to the presence of *small* components of very low frequencies in the motion. Even if this method failed, there is the alternative test of stopping either of the two forks when a 'Combinational' is instantly extinguished, whereas a 'simple' resonance is only abolished by stopping *one* of the two forks, and not *either*. One very characteristic feature which was noticed in the experiments was that while resonance-vibrations of the *summational* class were obtained with great ease up to fairly high orders and vigorously maintained, *differential* vibrations were not nearly so readily maintained, and it was found necessary, in order to realize them, to arrange matters so that none of the other resonances due to the forks, simple or summational, lay in the neighbourhood of the one sought for and could therefore extinguish it, the former being maintained by preference. The result noticed above is an inversion of the ordinary experience in acoustical work with combinational tones in which it is found that differentials are generally stronger and easier to demonstrate than summationals. The theoretical explanation of the effect will be discussed later in this paper.

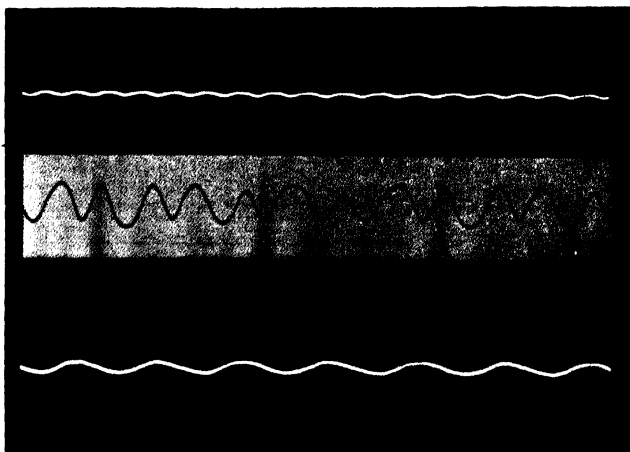


Fig. 1. SUMMATONAL MAINTENANCE: FREQUENCY $N_1 / 2 + N_2 / 2$.

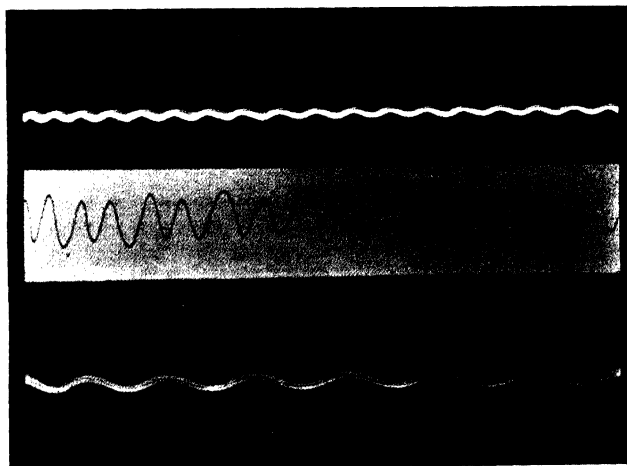


Fig. 2 SUMMATONAL MAINTENANCE: FREQUENCY $N_1 / 2 + n$

PHOTOGRAPHIC RECORD OF COMBINATIONAL VIBRATIONS.

In order to demonstrate that the frequency of maintenance is that given by the combinational formula referred to above, the method of vibration-curves was used. Arrangements were made to obtain simultaneous photographic records of the vibration of the two forks and of the maintained oscillation of the string. These records incidentally throw light on the *modus operandi* of the maintenance. The disposition of the apparatus employed is shown in the Diagram (page 6).

T_1 and T_2 are the two forks which stand with prongs vertical on the table. The string is stretched horizontally between the inner prongs of the forks as shown.

To enable its plane of vibration to be brought into the vertical at pleasure, the simple device described by me in a previous publication¹ is adopted. Each end of the string is attached to a loop of thread which is passed over the prong of the fork, instead of directly to the prong itself. The result of this mode of attachment is that the frequencies of vibration in the horizontal and vertical planes differ slightly, and this has the desired effect of keeping the vibration confined to the vertical if the tension of the string is suitably adjusted in each case. Immediately in front of the string is placed a camera, the plate-carrier of which has been removed, and which carries instead a square sheet fitted with a narrow vertical slit S as nearly as possible contiguous to the string. The light from an electric arc emerges from the nozzle of the lantern L ; and is then divided into three parts.

(1) One part passes first through the vertical slit S , then through the lens of the camera carrying it, and after suffering reflexion at the fixed mirror M passes on to the lens of the moving-plate camera DD (to be described below).

(2) The second part is deflected by the mirror M_1 , and after passing through a narrow horizontal slit S_1 suffers reflexion at the surface of a small plane mirror M_2 attached to the prong of the fork T_1 and is finally deflected by the mirror M_3 to the lens of the camera DD .

¹ A New Form of Melde's Experiment, Bulletin No. 6 of this Association, page 2.

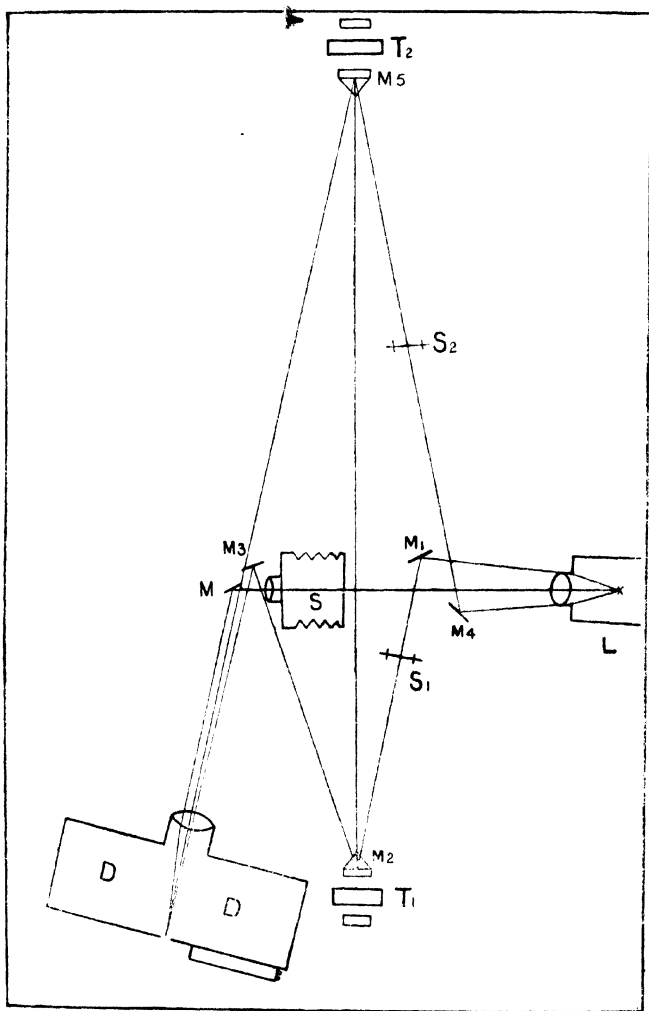


Diagram 1: Apparatus for the production and photographic record of Combinational Vibrations.

(3) The third part after deflexion by the mirror M_4 passes through the horizontal slit S_2 , and thence after reflexion at a plane mirror M_5 attached to the prong of the fork T_2 passes on to the camera DD .

By suitably adjusting the position of the slits S_1 and S_2 and the distance between the slit S and the lens of the camera carrying it, it was found possible without the use of any additional collimating lenses to obtain in the focal plane of the camera DD real images of the vertical slit S and of the horizontal slits S_2 and S_1 , the images of the latter being formed respectively above and below the images of the vertical slit S . A narrow vertical slit placed in the focal plane of the camera immediately in front of the ground-glass or its substitute, the photographic plate, cuts off everything except two brilliant spots of light, one for each fork, and between them the image of the illuminated vertical slit crossing which is seen the shadow of the string when at rest. The plate-holder can be moved (by hand) in horizontal grooves behind the slit in the focal plane, and when the forks are set in vibration we can thus obtain photographic records of simultaneous vibration-curves of the string and forks.

By counting the number of swings shown in each of the three curves appearing in a record, the relation between the frequencies of the forks and that of the maintained motion of the string can be observed and tested with the exact combinational formula $\frac{1}{2}(rN_1 \pm sN_2)$. Illustrations of this method will be given later.

ISOLATION OF INDIVIDUAL TYPES.

As already mentioned above, the vibrations of the summational class are obtained with great ease. Taking only the first four types of summationals, i.e., with $r = 1$ or 2 or 3 or 4 and $s = 1$ or 2 or 3 or 4, we 4×4 , i.e., 16 distinct types possible, or if we include the cases in which either r or s is zero, i.e., those of resonance due to each of the forks acting by itself, 24 distinct types. In practice summationals of types even higher than the fourth are distinguishable. For each of these various frequencies of vibration, the string may divide up into one, two, three or more ventral segments, according to circumstances. We have therefore a very large number of cases in which resonance is

possible, and it is a matter for considerable surprise that it should be at all possible (not to speak of its being quite easy) to isolate in experiment any one of these manifold modes and frequencies of vibration and obtain distinctive photographic records of the same. The explanation of this result is very instructive. It rests upon the following facts: for each of these summationals there is a limited and fairly well-defined range of frequency within which the natural frequency of the system should lie if maintenance is to be possible. If the natural frequency of the system lies within this range, the vibration is vigorously maintained. If outside it, the summational does not put in an appearance. The range becomes narrower and narrower as we go higher up the scale and is smallest in the very region where the summationals are relatively speaking most numerous. This effectually prevents their crowding in unduly upon each other.

The facts mentioned in the foregoing paragraph sufficiently explain the successful isolation of the several vibrational types. After a little practice it will be found easy to arrange that any given member of the series of summationals (if not of too high an order) is obtained and vigorously maintained. The necessary guide to the proper adjustment of tension is to be had by noticing the tensions at which the 'simple' resonances in various modes due to either of the forks acting alone occur, and by drawing up a table of frequencies of the summational vibrations it is a simple matter to get the right tension for any one of them.

In practical work, it will be found a useful device (besides adjusting the tension of the string to correspond with the frequency required), to regulate the amplitude of vibration of the forks in a suitable manner. This is readily done, if the forks are electrically maintained by altering the driving current or the position of the contact-maker. The formula to be borne in mind is, if r is large and s is small, to work the N_1 fork vigorously and the N_2 fork with quite a small amplitude of vibration: vice-versa if r is small and s is large. If r and s are to be nearly equal, the amplitudes are to be roughly commensurate with the values of r and s . This regulation of the amplitude ensures the desired summational being obtained without fail, and unaccompanied by other modes of vibration.

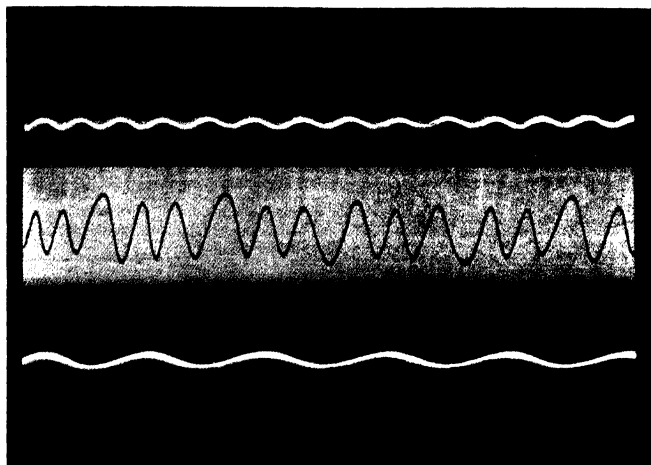


Fig. 1. SUMMATIOnAL MAINTENANCE: FREQUENCY $N_1 / 2 + 3 N_2 / 2$

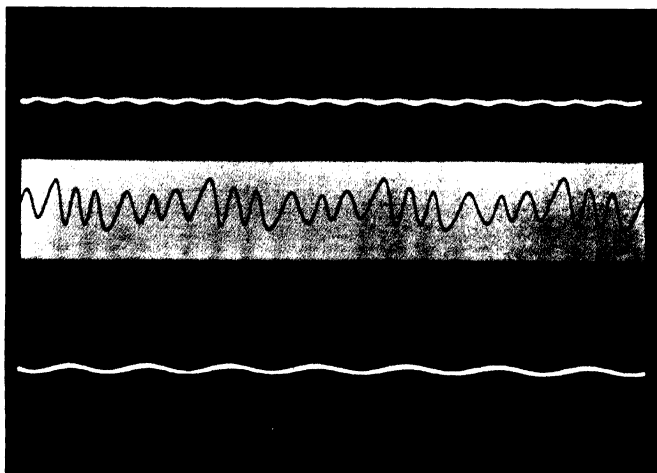


Fig. 2. SUMMATIOnAL MAINTENANCE; FREQUENCY $N_1 + N_2 / 2$.

VIBRATION-CURVES OF SUMMATIONALS.

Plate I to V exhibit the photographic records for all the nine summationals comprised within the first three types, for one summational of the fourth type and one of the fifth, i.e., eleven photographs in all. The two forks used had frequencies of 60 and 23·7 respectively per second. In the reproductions their vibration-curves appear white on a dark ground and that of the string dark on a bright background. The two former appear one on each side of the latter. It is obvious from an inspection of the vibration-curves of the string that in each case the principal part of the maintained motion is accompanied by subsidiary components. These components are introduced by the alteration of the character of the maintained motion due to the imposed variable spring, and it will be seen from the theoretical discussion to follow that they act as vehicles for the supply of energy to the system. In a few cases their periodicity is fairly evident to inspection.

Plate II, Fig. 1.—This is the first and most important summational, the frequency of maintenance being equal to the sum of half the frequencies of the forks. This is readily shown by counting the swings shown on each curve. Thus—

	Summational $\frac{1}{2} N_1 + \frac{1}{2} N_2$		
	Fork N_1	Fork N_2	String.
Number of swings ..	17·50	6·90	12·23
Calculated frequency ..	60·0	23·70	41·95
Observed frequency ..	60·0	23·70	41·93

An inspection of the vibration-curve of the string clearly shows the periodic flattening and sharpening of the maintained motion under the joint action of the components of the variable spring.

Plate I, Fig. 2.—This shows the next higher frequency of maintenance, summational $\frac{1}{2} N_1 + N_2$, frequency 53·7 approximately.

Plate II, Fig. 1, shows the summational $\frac{1}{2} N_1 + \frac{3}{2} N_2$, frequency 65·6.

Plate II, Fig. 2, shows the summational $N_1 + \frac{1}{2} N_2$, frequency 71·9.

Plate III, Fig. 1, shows the summational $N_1 + N_2$, frequency 83·7.

Plate III, Fig. 2, shows the summational $N_1 + \frac{3}{2} N_2$, frequency 95·6.

Plate IV, Fig. 1, shows the summational $\frac{3}{2} N_1 + \frac{1}{2} N_2$, frequency 101·9.

Plate IV, Fig. 2, shows the summational $\frac{3}{2} N_1 + N_2$ frequency 113·7.

Plate V, Fig. 1, shows the summational $\frac{3}{2} N_1 + \frac{3}{2} N_2$, frequency 125·6.

Plate V, Fig. 2, shows one of the summationals of the fourth type, frequency $2 N_1 + \frac{1}{2} N_2$, i.e., 131·9.

Plate V, Fig. 3, shows one of the summationals of the fifth type, frequency $2 N_1 + \frac{3}{2} N_2$, i.e., 179·3.

VIBRATION-CURVES OF DIFFERENTIALS.

Using the two forks of frequencies 60 and 23·7 it was not found possible to obtain any cases of differential resonances, as these lay in the region in which the primaries and summationals were present and were strongly maintained in preference. After some trial, however, using forks of adjustable frequencies with which the frequencies of possible differentials lay far removed from that of the stronger resonances due to either of the forks alone or their summationals, I succeeded in isolating two cases of differential resonance. Plate VI, Fig. 1, represents the differential of the first type, frequency $\frac{1}{2} N_1 - \frac{1}{2} N_2$ being 52·2, the frequencies of the two forks used, N_1 and N_2 , being respectively 128 and 23·7. Plate VI, Fig. 2, represents a differential of the second type, frequency $N_1 - N_2$ being 92·3, the frequencies of the two forks used, N_1 and N_2 , being respectively 128 and 35·7.

It will be seen that in both of these cases, the frequency of the differential is such that it cannot be readily confused with that of any resonances due to either of the forks acting alone or to their summationals.

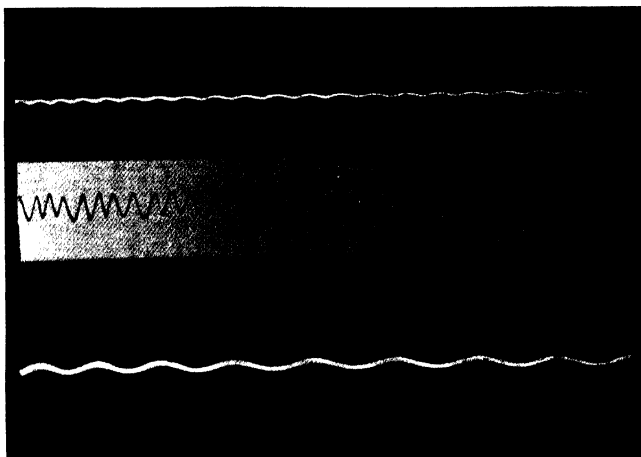


Fig. 1. SUMMATIOnAL MAINTENANCE: FREQUENCY $N_1 + N_2$.

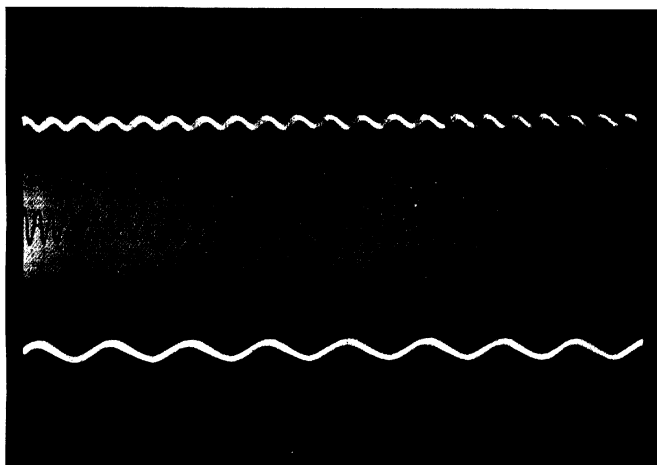


Fig. 2. SUMMATIOnAL MAINTENANCE: FREQUENCY $N_1 + 3 N_2 / 2$.

GENERAL THEORY OF COMBINATIONAL MAINTENANCE.

The equation of motion of a simple oscillatory system having one degree of freedom when subject to two periodic forces varying its spring may be written as

$$\ddot{U} + k\dot{U} + n^2U = 2U[a_1 \sin 2p_1t + a_2 \sin 2p_2t + \beta_1 \cos 2p_1t + \beta_2 \cos 2p_2t] \quad (1)$$

$$= U[2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2)] \quad (2)$$

It may also be written in the form

$$\ddot{U} + k\dot{U} + \left(n^2 - 2\gamma_1 \sin \overline{2p_1t + E_1} - 2\gamma_2 \sin \overline{2p_2t + E_2} \right) U = 0 \quad (3)$$

It is instructive to compare the equation of motion as it is written in forms (1) and (2) above, with that of an asymmetrical system subject to double forcing considered by Helmholtz. The latter may be written as

$$\ddot{U} + k\dot{U} + (n^2 - \gamma U)U = 2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2) \quad (4)$$

or as

$$\ddot{U} + k\dot{U} + n^2U = \gamma U^2 + 2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2) \quad (5)$$

Equations (2) and (5) are analogous in so far as the term on the right-hand side which represents the "disturbing force" acting on the system is in both cases a function of two variables, i.e., the time, and the configuration of the system: the form of the function is however different in the two cases.

Equations (3) and (4) are analogous in so far as that the coefficient of the third term on the left which represents the "spring" of the system is in both cases a variable. But here the analogy ends, for in one case the variable part of the spring is an independent function of the time, in the other case it is a function of the configuration only.

In any case, however, there is abundant material to suggest that the maintenance of a series of combinational vibrations should be possible under the joint action of two periodic forces of different frequencies varying the spring, and this is fully verified by the results of the experiments described above.

DISCUSSION OF DIAGRAM OF PERIODICITIES.

The exact process by which the maintenance of the combinational vibrations is effected in these experiments is best under-

stood by analogy with the simpler case of oscillations under variable spring of only one periodicity, and by reference to diagram 2 below. In the theory of oscillations maintained by a simple variable spring (Bulletin No. 6, Section IV), it was shown that when resonance was secured by adjusting the natural frequency of the system to any multiple of half the frequency of the impressed force, the principal part of the maintained motion and the subsidiary components of smaller amplitudes introduced under the action of the variable spring could be arranged in the form of a Fourier series. For example, in the case of the 4th type of maintenance, the equation of motion is

$$\ddot{U} + k\dot{U} + n^2U = 2aU \sin 2pt,$$

and n being nearly equal to $4p$, we have as the solution

$$U = A_2 \sin 2pt + A_4 \sin 4pt + \text{etc.} \\ + B_0 + B_2 \cos 2pt + B_4 \cos 4pt + \text{etc.}$$

Similarly in the case of the 3rd type of maintenance where n is nearly equal $3p$ we have

$$U = A_1 \sin pt + A_3 \sin 3pt + \text{etc.} \\ + B_1 \cos pt + B_3 \cos 3pt + \text{etc.}$$

It will thus be seen that in each case, the subsidiary components introduced under the action of the variable spring proceed by successive differences of $2pt$. The components having smaller frequencies than that of the system were the vehicles for the supply of energy required for the maintenance of the oscillations: those having higher frequencies play no such part, but are equivalent merely to a small alteration in the natural frequency of the system. The components of smaller frequencies are none of them negligible so far as the explanation of the maintenance is concerned: those of higher frequencies can generally (though not always) be safely ignored.

In diagram 2 each of the points marked corresponds to a frequency of the system at which (to a close approximation) resonance is possible under the sole or joint action of the two components of variable spring in the experiments described above. The diagram also enables us to see at a glance the periodicity of

the subsidiary components in the motion in each such case. For, starting at the resonance-point for the case, we move by successive steps of $2p_1$ and $2p_2$ respectively parallel to the two axes, and each point so arrived at represents a component in the motion induced under the action of the variable spring.

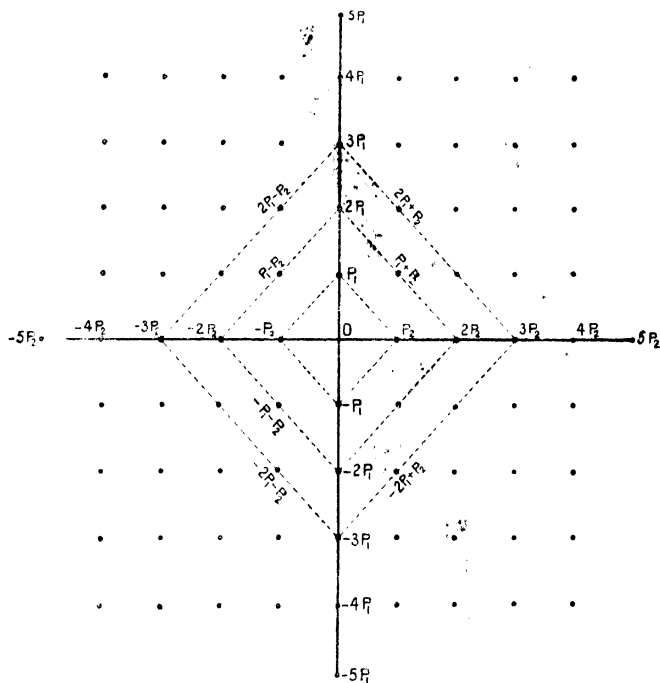


Diagram 2: Representation of Periodicities of Combinational Maintenance.

Only about one-fourth of the total number of points in the diagram can be arrived at in this way from a given starting-point. Such of the components as are represented by points lying outside the diagonal square drawn through the resonance-point, (as in the specimens shown by dotted lines in the diagram), may be neglected in considering the maintenance of the motion. For

example, when considering the case $n=(p_1+p_2)$ nearly, only the components which are trigonometrical functions of $(p_1+p_2)t$ and $(p_1-p_2)t$ need be taken into account to arrive at an approximate solution. It will be seen that alternative paths starting from the same point lead to periodicities having the same value, with or without the sign reversed; and it is this fact and the negligibility of points lying outside the diagonal square in each case that enables a definite solution of the equation of motion to be obtained easily by the method of approximations.

The diagram of periodicities has other uses. It enables us to obtain at a glance an indication of the circumstances under which maintenance can be successfully effected in each case, and to arrange the experimental conditions accordingly. For instance, in the case of motion under one of the components of variable spring alone, we know that the adjustment of the natural frequency of the system with reference to that of the impressed force should be more and more accurate as we proceed outwards along the axes of the diagram and meet the successive resonance-points. In the case of the point p_1 the adjustment should be accurate to the order α , in the case of the point $2p_1$ to the order α^2 , and so on, α being the coefficient of the variable spring (supposed small). The diagram suggests that a similar increase in the accuracy of adjustment would be found necessary as we proceed outwards from the origin in any other direction and meet resonance-points situated on successive diagonals (shown as dotted lines). This indication is amply confirmed both by experiment and by the detailed mathematical treatment. For instance it will be shown below that a degree of adjustment accurate to the order α^3 would be necessary for the point p_1+p_2 which lies midway between $2p_1$ and $2p_2$.

The position of any resonance-point on the diagram is also found to indicate approximately the amplitude of vibration of the two tuning-forks required for successful maintenance.

THEORY OF SUMMATONAL OF THE FIRST TYPE.

We now proceed to consider the detailed theory of the simplest summational type in which the frequency of maintenance is equal to the sum of half the frequencies of the forks. As

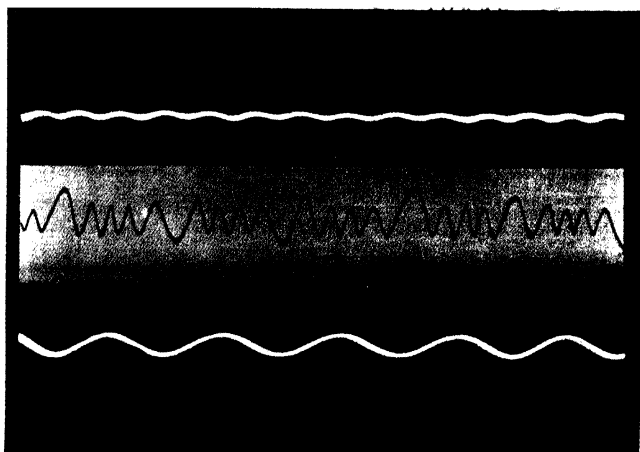


Fig. 1 SUMMATONAL MAINTENANCE: FREQUENCY $3 N_1 / 2 + N_2$

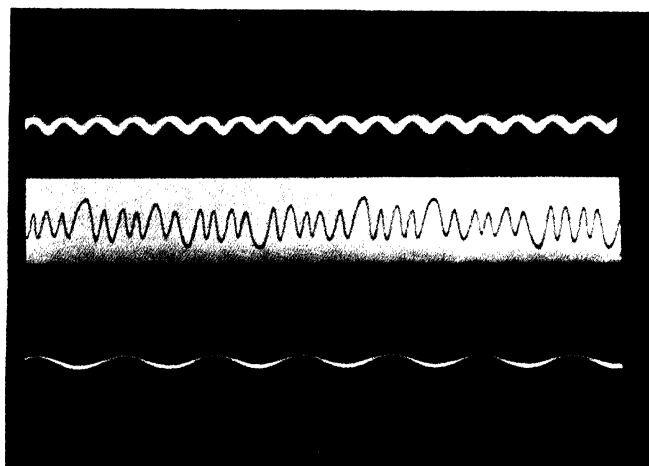


Fig. 2 SUMMATONAL MAINTENANCE: FREQUENCY $3 N_1 / 2 + N_2$

already explained, the whole of the string behaves as one unit, in other words, we may write down the equation of motion as that of a system having only one degree of freedom. The equation is

$$\ddot{U} + k\dot{U} + n^2U = 2U[a_1 \sin 2p_1t + a_2 \sin 2p_2t + \beta_1 \cos 2p_1t + \beta_2 \cos 2p_2t] \quad (1)$$

Since by assumption n is nearly equal to $p_1 + p_2$, we may commence building up a solution by putting

$$U = A_1 \sin (p_1 + p_2)t + B_1 \cos (p_1 + p_2)t + \text{etc.} + \text{etc.} \quad (6)$$

On substituting the first two terms on the right of (6) for U in the right-hand side of equation (1) and expanding the terms in it, we find that none of them is a sine or cosine of $(p_1 + p_2)t$, i.e. represents a force which is competent to excite resonance, if we regard (1) as the ordinary equation of forced oscillations. It is obvious, therefore, that to solve the equation even approximately and explain the maintenance of the motion, we have to take a second subsidiary pair of terms in the expression for U . The frequency of these terms is ascertained at once by reference to the diagram of periodicities given above, and we may then write

$$U = A_1 \sin (p_1 + p_2)t + B_1 \cos (p_1 + p_2)t \\ + A_2 \sin (p_1 - p_2)t + B_2 \cos (p_1 - p_2)t + \text{etc.} \quad (7)$$

The terms in A_2 and B_2 are of course small compared with those in A_1 B_1 , yet are sufficiently large to make their presence felt in the vibration-curve pictured previously. They are introduced by the action of *either* of the two periodic components of the variable spring on the fundamental motion, and in their turn maintain the latter by making the requisite supply of energy to the system possible. This can be shown by writing out the equation of work in a manner similar to that adopted in my paper in the *Philosophical Magazine* for October 1912. In the present case it can be shown in a much simpler way by merely equating separately the terms of various periodicities on either side of equation (1) after substituting the value of U given by (7).

We have thus :

$$\begin{aligned} (\text{writing } n^2 - (p_1 + p_2)^2 = \theta_1 \text{ and } n^2 - (p_1 - p_2)^2 = \theta_2, k(p_1 + p_2) = \phi_1 \\ \text{and } k(p_1 - p_2) = \phi_2 \text{ for brevity,} \end{aligned}$$

$$\begin{aligned}
\theta_1 A_1 - \phi_1 B_1 &= A_2(\beta_2 - \beta_1) + B_2(\alpha_1 + \alpha_2) \\
\phi_1 A_1 + \theta_1 B_1 &= A_2(\alpha_1 - \alpha_2) + B_2(\beta_1 + \beta_2) \\
\theta_2 A_2 - \phi_2 B_2 &= A_1(\beta_2 - \beta_1) + B_1(\alpha_1 - \alpha_2) \\
\phi_2 A_2 + \theta_2 B_2 &= A_1(\alpha_1 + \alpha_2) + B_1(\beta_1 + \beta_2)
\end{aligned} \tag{8}$$

It will be seen that these four equations were derived by retaining only terms containing trigonometrical functions of $(p_1 + p_2)t$ and $(p_1 - p_2)t$ and neglecting all others. Before considering the effect, if any, of the neglected terms, it is well to discuss the physical significance of the equations. They give us the three ratios A_1, B_1, A_2, B_2 in terms of the known quantities $\theta_1, \theta_2, \phi_1, \phi_2$, and $\alpha_1, \alpha_2, \beta_1, \beta_2$, as an approximate solution of the equation of motion and leave us in addition a relation between these 'constants' which must be satisfied for steady motion to be possible.

We now proceed to solve the equations. This may, of course, be done in the usual way in terms of certain determinants, but it is more instructive to proceed by an approximate method retaining only terms up to the order of smallness desired. Assuming that $\alpha_1, \alpha_2, \beta_1, \beta_2$ (the coefficients of variable spring) are small, it is obvious from the equations that A_2, B_2 are small compared with A_1, B_1 . Further ϕ_2 is a very small quantity compared with θ_2 (which is finite and large), since the former is proportional to k , the coefficient of friction, which must itself be small for maintenance to be possible. We are therefore justified in neglecting ϕ_2 and writing the last two of equations (8) thus:—

$$\begin{aligned}
\theta_2 A_2 &= A_1 (\beta_2 - \beta_1) + B_1 (\alpha_1 - \alpha_2) \\
\theta_2 B_2 &= A_1 (\alpha_1 + \alpha_2) + B_1 (\beta_1 + \beta_2)
\end{aligned} \tag{9}$$

These equations give us the subsidiary components of motion A_2, B_2 , in terms of the principal parts A_1, B_1 . Substituting these values in the first two of equations (8), we have

$$\begin{aligned}
\theta_1 A_1 - \phi_1 B_1 &= \frac{A_1}{\theta_2} \left[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + 2(\alpha_1 \alpha_2 - \beta_1 \beta_2) \right] \\
&\quad + \frac{B_1}{\theta_2} \left[2(\alpha_1 \beta_2 + \alpha_2 \beta_1) \right] \\
\phi_1 A_1 + \theta_1 B_1 &= \frac{A_1}{\theta_2} \left[2(\alpha_1 \beta_2 + \alpha_2 \beta_1) \right]
\end{aligned}$$

$$+ \frac{B_1}{\theta_1} \left[(a_1^2 + \beta_1^2) + (a_2^2 + \beta_2^2) - 2(a_1 a_2 - \beta_1 \beta_2) \right] \quad (10)$$

Writing

$$\begin{aligned} \theta_1 - \frac{1}{\theta_2} \left[(a_1^2 + \beta_1^2) + (a_2^2 + \beta_2^2) \right] &= \theta \\ \frac{2}{\theta_2} (a_1 a_2 - \beta_1 \beta_2) &= a \\ \frac{2}{\theta_2} (a_1 \beta_2 + a_2 \beta_1) &= b \end{aligned}$$

we have

$$\begin{aligned} (\theta - a)A_1 &= (\phi_1 + b)B_1 \\ (\theta + a)B_1 &= -(\phi_1 - b)A_1 \end{aligned} \quad (11)$$

The solution of this is

$$\frac{B_1}{A_1} = \frac{\theta - a}{\phi_1 + b} = \frac{b - \theta}{\theta + a} \quad (12)$$

and the eliminant, i.e., the relation between the constants involved which must be satisfied for maintenance to be possible, is

$$\theta^2 - a^2 = b^2 - \phi_1^2 \quad (13)$$

It is interesting to consider a few special cases. For instance put

$$\beta_1 = \beta_2 = 0 \text{ and } a_1 = a_2 = a/2.$$

From (9) we find then that

$$A_2 = 0 \text{ and } B_2 = A_1 a/2.$$

Also

$$\theta_1 + \frac{\phi_1^2}{\theta_1} = a^2/\theta \quad (14)$$

From (14) it is evident that θ_1 and ϕ_1 are both of the order a^2/θ , i.e., both the friction and adjustment of frequency for resonance must be correct to the second order of small quantities. The equations leave the actual amplitude of the motion indeterminate. In practice both the necessary adjustment of frequency and the determinateness of the amplitude would be secured by the fact that n is not absolutely constant, in other words by the variation of spring existing in free oscillations of sensible amplitude. It will be seen that the term B_2 is small compared with A_1 , since the variable part of the spring is small compared with the permanent spring: nevertheless the term B_2 plays a very important part

in the maintenance of the motion, being, as is evident from the foregoing equations, the vehicle for the supply of the requisite energy to the system.

It will be seen that the equations cannot be satisfied if $\theta_1 = 0$, as $\theta_1 + \frac{\phi_1^2}{\theta_1}$ then becomes infinitely large. According to these equations therefore, resonance is possible only when θ_1 has a small but a definite positive value: i.e., when the frequency of the *free* oscillations is very slightly greater than the sum of the half-frequencies of the two imposed variations of spring. This conclusion would no doubt have to be modified in view of two factors which we have not so far taken into account. First, the neglected terms in the motion which are trigonometrical functions of $(p_1 + 3p_2)t$ and $(3p_1 + p_2)t$ etc., etc. The two terms $(p_1 + 3p_2)t$, $(3p_1 + p_2)t$ which are of the order α when compared with the fundamental motion $(p_1 + p_2)t$ cannot actually assist in maintaining it. This can be shown from very simple considerations. For one thing, they are not introduced by the action of *both* the components of variable spring. The first is due to one component and the second is due to the other. The components in the restoring force due to their action and which are trigonometrical functions of $(p_1 + p_2)t$ are in such a *phase* that their effect is equivalent to a small increase in the frequency of free oscillations of the system which is some importance when θ_1 is very small.

The second factor which we have to take into account is the variation of spring existing in free oscillations of large amplitude. This, as in the case of oscillations maintained by a single variable spring, when expanded is found to contain both constant and periodic terms. The former are equivalent to a direct increase in the natural frequency of the system, and the latter profoundly modify the effect of the impressed forces and the phases of the respective components in the motion when the amplitude is at all sensible.

THEORY OF SUMMATIONALS OF THE SECOND AND HIGHER TYPES.

As typical of the summationals of the second type we may take the case $n_1 = 2p_1 + p_2$ (nearly). The equation of motion is

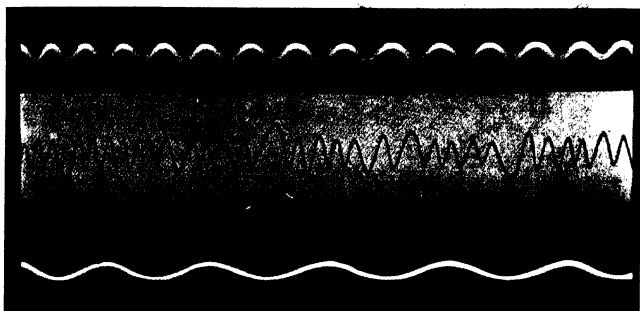


Fig. 1. SUMMATIIONAL MAINTENANCE : FREQUENCY $3 N_1 / 2 + 3 N_2 / 2$.

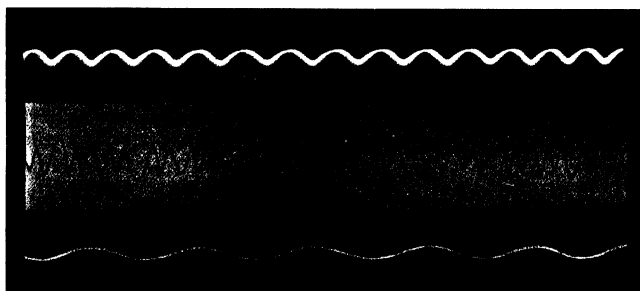


Fig. 2. SUMMATIIONAL MAINTENANCE : FREQUENCY $2 N_1 + N_2 / 2$.

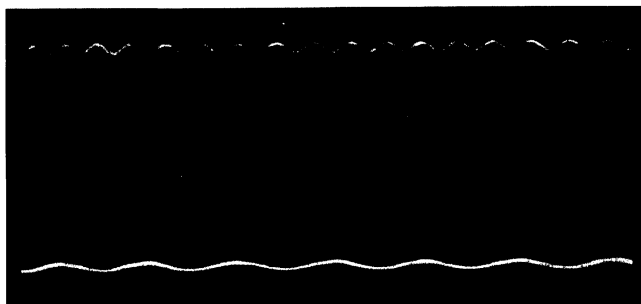


Fig. 3. SUMMATIIONAL MAINTENANCE : FREQUENCY $2 N_1 + 5 N_2 / 2$.

$$\ddot{U} + k\dot{U} + n^2U = 2U \left[\begin{array}{l} \alpha_1 \sin 2p_1t + \alpha_2 \sin 2p_2t \\ + \beta_1 \cos 2p_1t + \beta_2 \cos 2p_2t \end{array} \right] \quad (1)$$

We may to start with put

$$U = A_1 \sin (2p_1 + p_2)t + B_1 \cos (2p_1 + p_2)t \\ + \text{etc.} + \text{etc.} \quad (15)$$

As before, we can only solve the equation by taking certain additional terms on the right of (15). A reference to the diagram of periodicities shows that we have to take three additional pairs of terms to get an approximate solution. We may thus write

$$U = A_1 \sin (2p_1 + p_2)t + B_1 \cos (2p_1 + p_2)t \\ + A_2 \sin (2p_1 - p_2)t + B_2 \cos (2p_1 - p_2)t \\ + A_3 \sin p_2t + B_3 \cos p_2t \\ + A_4 \sin 3p_2t + B_4 \cos 3p_2t \quad (16)$$

Of course, the terms in A_1 and B_1 are the largest in amplitude. In substituting the right-hand side of (16) for U in equation (1) and writing down in the results, we may use the following abbreviations:

$$n^2 - (2p_1 + p_2)^2 = \theta_1, \quad k(2p_1 + p_2) = \phi_1 \\ n^2 - (2p_1 - p_2)^2 = \theta_2, \quad k(2p_1 - p_2) = \phi_2 \\ n^2 - p_2^2 = \theta_3, \quad kp_2 = \phi_3 \\ n^2 - 9p_2^2 = \theta_4, \quad 3kp_2 = \phi_4 \quad (17)$$

By equating the various sine and cosine terms on either side of (1) after substitution, we have

$$\begin{aligned} \theta_1 A_1 - \phi_1 B_1 &= \beta_1 A_3 + \alpha_1 B_3 + \beta_2 A_2 + \alpha_2 B_2 \\ \phi_1 A_1 + \theta_1 B_1 &= -\alpha_1 A_3 + \beta_1 B_3 - \alpha_2 A_2 + \beta_2 B_2 \\ \theta_2 A_2 - \phi_2 B_2 &= \beta_2 A_1 - \alpha_2 B_1 - \beta_1 A_3 + \alpha_1 B_3 \\ \phi_2 A_2 + \theta_2 B_2 &= \alpha_2 A_1 + \beta_2 B_1 + \alpha_1 A_3 + \beta_1 B_3 \\ \theta_3 A_3 - \phi_3 B_3 &= \beta_1 A_1 - \alpha_1 B_1 - \beta_1 A_2 + \alpha_1 B_2 \\ &\quad + \beta_2 A_4 - \alpha_2 B_4 \\ \phi_3 A_3 + \theta_3 B_3 &= \alpha_1 A_1 + \beta_1 B_1 + \alpha_1 A_2 + \beta_2 B_2 \\ &\quad + \alpha_2 A_4 + \beta_2 B_4 \\ \theta_4 A_4 - \phi_4 B_4 &= \beta_1 A_3 + \alpha_1 B_3 \\ \phi_4 A_4 + \theta_4 B_4 &= -\alpha_2 A_3 + \beta_2 B_3 \end{aligned} \quad (18)$$

From the last four equations in (18) it will be seen that a further simplification can be effected. For A_4 and B_4 are of the order α_2, β_2 in comparison with A_3 and B_3 , and the terms $\beta_2 A_4$, $\alpha_2 B_4$, $\alpha_2 A_4$ and $\beta_2 B_4$ on the right-hand side of the fifth and sixth equations in (18) can therefore be neglected in comparison with all other terms involved. We are therefore finally left with six equations only

$$\begin{aligned}\theta_1 A_1 - \phi_1 B_1 &= \beta_1 A_2 + \alpha_1 B_2 + \beta_2 A_2 + \alpha_2 B_2 \\ \phi_1 A_1 + \theta_1 B_1 &= -\alpha_1 A_3 + \beta_1 B_3 - \alpha_2 A_2 + \beta_2 B_2 \\ \theta_1 A_2 - \phi_2 B_2 &= \beta_2 A_1 - \alpha_2 B_1 - \beta_1 A_3 + \alpha_1 B_3 \\ \phi_1 A_2 + \theta_2 B_2 &= \alpha_2 A_1 + \beta_2 B_1 + \alpha_1 A_3 + \beta_1 B_3 \\ \theta_2 A_3 - \phi_3 B_3 &= \beta_1 A_1 - \alpha_1 B_1 - \beta_1 A_2 + \alpha_1 B_2 \\ \phi_2 A_3 + \theta_3 B_3 &= \alpha_1 A_1 + \beta_1 B_1 + \alpha_1 A_2 + \beta_1 B_2\end{aligned}\quad (19)$$

These equations contain only terms having three periodicities, i.e. the principal part of the maintained motion of frequency $(2p_1 + p_2)/2\pi$ and two others, subsidiary to it, which are given by the two nearest admissible points on the periodicity diagram, i.e., of frequencies $(2p - p_2)/2\pi$ and $p_2/2\pi$ respectively. These components are derived from the principal motion by the action on it of the variations in spring, and serve to maintain it permanently in the presence of dissipative forces. They are both small compared with the main motion provided $\alpha_1, \alpha_2, \beta_1, \beta_2$ are small. Plate II, Fig. 2, represents the vibration curve of this type of summational, and the component of frequency $p_2/2\pi$ (i.e. half that of the graver fork) is very obvious to inspection.

Equations (19) when solved give us values for the five ratios $\frac{B_1}{A_1}, \frac{B_2}{A_1}, \frac{B_3}{A_1}, \frac{A_2}{A_1}$ and $\frac{A_3}{A_1}$ and leave us in addition a relation between the "constants" involved in the equations which must be satisfied for steady maintenance to be possible. The solution of the equations by the method of determinants is really a formidable business and is in fact unnecessary: an approximate method of solving them may be used which gives results quite as accurate as the equations themselves. We notice that A_2, B_2, A_3, B_3 are small quantities relatively to A_1, B_1 , and further, since k , the coefficient of friction, is necessarily very small for maintenance to be

possible, the quantities ϕ_2, ϕ_3 are negligible in comparison with θ_2 and θ_3 . The first step in solving the equations (19) is therefore to simplify the last four of them and obtain approximate values for A_2, B_2, A_3, B_3 .

We thus have

$$\begin{aligned}\theta_2 A_2 &= \beta_2 A_1 - \alpha_2 B_1 + \frac{1}{\theta_2} \left(\overline{\alpha_1^2 - \beta_1^2} A_1 + 2\alpha_1 \beta_1 B_1 \right) \\ \theta_2 B_2 &= \alpha_2 A_1 + \beta_2 B_1 + \frac{1}{\theta_2} \left(2\alpha_1 \beta_1 A_1 - \overline{\alpha_1^2 - \beta_1^2} B_1 \right) \\ \theta_3 A_3 &= \beta_3 A_1 - \alpha_3 B_1 + \frac{1}{\theta_3} \left(\overline{\alpha_1 \alpha_2 - \beta_1 \beta_2} A_1 + \overline{\alpha_1 \beta_2 + \alpha_2 \beta_1} B_1 \right) \\ \theta_3 B_3 &= \alpha_3 A_1 + \beta_3 B_1 + \frac{1}{\theta_3} \left(\overline{\alpha_1 \beta_2 + \alpha_2 \beta_1} A_1 - \overline{\alpha_1 \alpha_2 - \beta_1 \beta_2} B_1 \right) \quad (20)\end{aligned}$$

The equations give us the values of the subsidiary components of motion in terms of the principal part and of the coefficients of the variable spring by which they are produced.

Substituting these values in the first two of equations (19) we have

$$\begin{aligned}\theta_1 A_1 + \phi_1 B_1 &= \frac{A_1}{\theta_2 \theta_3} \left[\theta_2 (\alpha_1^2 + \beta_1^2) + \theta_3 (\alpha_2^2 + \beta_2^2) + 2\beta_2 (\alpha_1^2 - \beta_1^2) + 4\alpha_2 \alpha_1 \beta_1 \right] \\ &\quad + \frac{B_1}{\theta_2 \theta_3} \left[4\beta_2 \alpha_1 \beta_1 - 2\alpha_2 (\alpha_1^2 - \beta_1^2) \right] \\ \phi_1 A_1 + \theta_1 B_1 &= \frac{A_1}{\theta_2 \theta_3} \left[4\beta_2 \alpha_1 \beta_1 - 2\alpha_2 (\alpha_1^2 - \beta_1^2) \right] \\ &\quad + \frac{B_1}{\theta_2 \theta_3} \left[\theta_2 (\alpha_1^2 + \beta_1^2) + \theta_3 (\alpha_2^2 + \beta_2^2) - 2\beta_2 (\alpha_1^2 - \beta_1^2) - 4\alpha_2 \alpha_1 \beta_1 \right] \quad (21)\end{aligned}$$

Writing

$$\begin{aligned}\theta_1 - \frac{1}{\theta_2 \theta_3} \left[\theta_2 (\alpha_1^2 + \beta_1^2) + \theta_3 (\alpha_2^2 + \beta_2^2) \right] &= \theta \\ \frac{1}{\theta_2 \theta_3} \left[2\beta_2 (\alpha_1^2 - \beta_1^2) + 4\alpha_2 \alpha_1 \beta_1 \right] &= a \\ \frac{1}{\theta_2 \theta_3} \left[4\beta_2 \alpha_1 \beta_1 - 2\alpha_2 (\alpha_1^2 - \beta_1^2) \right] &= b,\end{aligned}$$

equations (21) may be put into the form

$$\begin{aligned}(\theta - a)A_1 &= (\phi_1 + b)B_1 \\ (\theta + a)B_1 &= -(\phi_1 - b)A_1 \quad (22)\end{aligned}$$

It will be seen that equations (22) are identical in *form* with equations (II) obtained for the summational of the first type and the further discussion must proceed on much the same lines.

The solution is

$$\frac{B_1}{A_1} = \frac{\theta - a}{\phi_1 + b} = \frac{b - \phi}{\theta + a} \quad (23)$$

and the eliminant is

$$\theta^2 - a^2 = b^2 - \phi_1^2 \quad (24)$$

We may as before consider the special case in which $\beta_1 = \beta_2 = 0$ and $a_1 = a_2 = a$

$$\begin{aligned} \theta_2 A_2 &= -a B_1 + a^2 A_1 / \theta_3 \\ \theta_2 B_2 &= a A_1 - a^2 B_1 / \theta_3 \\ \theta_1 A_3 &= -a B_1 + a^2 A_1 / \theta_3 \\ \theta_1 B_3 &= a A_1 - a^2 B_1 / \theta_3 \\ \frac{B_1}{A_1} &= \frac{\theta_1 \theta_2 \theta_3 - a^2 (\theta_1 + \theta_2)}{\phi_1 \theta_2 \theta_3 - 2a^2} = \frac{-2a^2 - \phi_1 \theta_2 \theta_3}{\theta_1 \theta_2 \theta_3 - a^2 (\theta_2 + \theta_3)} \\ [\theta_1 \theta_2 \theta_3 - a^2 (\theta_1 + \theta_2)]^2 &= 4a^6 - \phi_1^2 \theta_2^2 \theta_3^2 \end{aligned} \quad (25)$$

For maintenance to be theoretically possible in this case, the frictional coefficient k should be of the third order of small quantities and the adjustment of frequency must therefore be accurate up to the same order.

Cases of summationals of higher orders can be worked out in a similar manner, the approximation being carried to a higher and higher degree as we rise up the series.

THEORY OF DIFFERENTIALS.

The solution of the equation of motion in the case of the first differential is obviously to be obtained in this case by merely writing $A_1, B_1, \theta_1, \phi_1$ for $A_2, B_2, \theta_2, \phi_2$ and vice versa, in equations (8) obtained for the summational of the first type. We thus have

$$\begin{aligned} \theta_1 A_1 - \phi_1 B_1 &= A_2 (\beta_2 - \beta_1) + B_2 (a_1 - a_2) \\ \phi_1 A_1 + \theta_1 B_1 &= A_2 (a_1 + a_2) + B_2 (\beta_1 + \beta_2) \\ \theta_2 A_2 - \phi_2 B_2 &= A_1 (\beta_2 - \beta_1) + B_1 (a_1 + a_2) \\ \phi_2 A_2 + \theta_2 B_2 &= A_1 (a_1 - a_2) + B_1 (\beta_1 + \beta_2) \end{aligned} \quad (26)$$

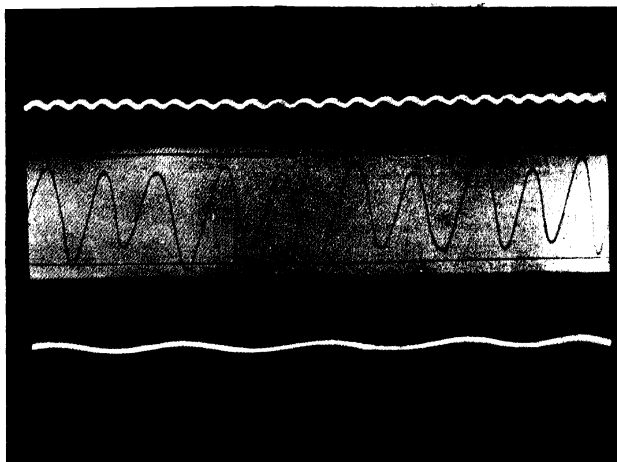


Fig. 1. DIFFERENTIAL MAINTENANCE; FREQUENCY $N_1 / 2 - N_2 / 2$.

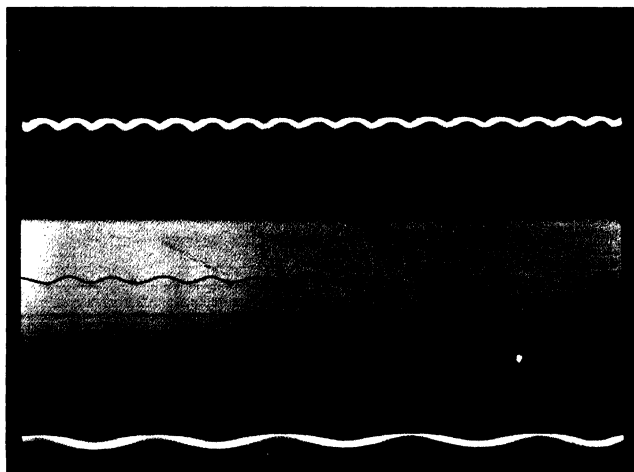


Fig. 2. DIFFERENTIAL MAINTENANCE; FREQUENCY $N_1 - N_2$.

where

$$U = A_1 \sin (\phi_1 - \phi_2)t + B_1 \cos (\phi_1 - \phi_2)t \\ + A_2 \sin (\phi_1 + \phi_2)t + B_2 \cos (\phi_1 + \phi_2)t + \text{etc.} \quad (27)$$

and

$$\dot{\phi}_1 = n^2 - \overline{\phi_1 - \phi_2}^2, \quad \dot{\phi}_2 = n^2 - \overline{\phi_1 + \phi_2}^2 \\ \phi_1 = k(\phi_1 - \phi_2) \quad \& \quad \phi_2 = k(\phi_1 + \phi_2)$$

Proceeding as before, we put

$$\theta A = A_1(\beta_2 - \beta_1) + B_1(a_1 + a_2) \\ \theta_2 B = A_1(a_1 - a_2) + B_1(\beta_1 + \beta_2) \quad (28) \\ \phi_1 A_1 - \phi_2 B_1 = \frac{A_1}{\theta} \left[(a_1^2 + \beta_1^2) + (a_2^2 + \beta_2^2) - 2(a_1 a_2 + \beta_1 \beta_2) \right] \\ + \frac{B_1}{\theta_2} \left[2(a_1 \beta_2 - a_2 \beta_1) \right] \\ \phi_2 A_1 + \theta_1 B_2 = \frac{A_1}{\theta} \left[2(a_1 \beta_2 - a_2 \beta_1) \right] \\ + \frac{B_1}{\theta_2} \left[(a_1^2 + \beta_1^2) + (a_2^2 + \beta_2^2) + 2(a_1 a_2 + \beta_1 \beta_2) \right] \quad (29)$$

It will be seen that equations (29) are of the same general form as (10) with certain modifications. The terms on the right-hand sides of (10) and (29) can be derived from each other by writing a_2 for $-a_2$ and vice versa. The further reduction of equations (29) may be proceeded with in the usual way.

The outstanding feature of the differential types is the relative difficulty of isolating and maintaining them successfully. No doubt this is partly due to the fact that the differentials of any given order are of much lower frequency than the summationals of the same order, and they lie therefore generally in the very region of frequencies in which simpler types of resonance are maintained far more powerfully over wide ranges. These latter are maintained by preference and extinguish the differentials. The foregoing however does not appear to be a complete explanation. Possibly the following further considerations must also be taken into account in explaining the relative poverty of differentials. In the mathematical discussion, it was shown that the subsidiary components of motion introduced under the action of

the variable spring, themselves enabled the principal motion to be maintained, and the relative amplitudes and phases of these components were determined on the assumption that n^2 , the free spring of the system, was a constant. It was however indicated that in practice this was not strictly the case, and in fact the success or otherwise of the experiments, i.e. the steady maintenance of vibration in a certain amplitude, is dependent on the quantity n^2 not being itself absolutely a constant. For large displacements n^2 is greater than for small displacements and the equation of motion when modified to take account of this fact may be written thus,

$$\ddot{U} + kU + \left[n^2 + mU^2 - 2\alpha_1 \sin 2p_1 t + \beta_1 \cos 2p_1 t - 2\alpha_2 \sin 2p_2 t + \beta_2 \cos 2p_2 t \right] U = 0 \quad (30)$$

The quantity mU^2 has been added to the third term within the brackets to represent the increase of spring for large displacements in a *symmetrical* system. It is obvious that when expanded for any periodic solution of U , mU^2 will give us both constant and periodic terms. The latter, i.e. the periodic terms, would be of various frequencies, and of them the most important would be those which are sines or cosines of $2p_1 t$ or $2p_2 t$, since they would directly modify the action of the components of the variable spring. It is therefore quite evident that the amplitudes and phases of the subsidiary components of motion would not be the same as when mU^2 is omitted. Some components would tend to increase at the expense of others. Instances of such action have already been furnished in Bulletin No. 6 (page 28) when discussing maintenance by a simple variation of spring.

What we may expect to find is that when mU^2 is taken into account in the equation of motion, the subsidiary components which maintain differentials are less effective than they would otherwise be, whereas, in the case of summationals, they would be more effective. For, in the former case, some of them at least are of frequency higher than that of the maintained motion, in the latter they are invariably less, and the components of lower frequencies are encouraged at the expense of those of higher frequencies.

II. On Motion in a Periodic Field of Force.

By C. V. Raman, M.A.

(Plates VII to XIV.)

SUMMARY OF CONTENTS.

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VIBRATIONS MAINTAINED BY A PERIODIC FIELD OF FORCE.

The experimental study of the motion of a dynamical system in a periodic field of force leads to results of quite exceptional interest. One aspect of the problem, i.e. the oscillatory motion of the system about a position of equilibrium in the field, has some affinities to the case of vibrations maintained by a variable spring which I have dealt with in my previously published work, but the two classes of investigations lead to results which differ from one another, yet are related in a most remarkable way. By experimenting on stretched strings subjected to a variable tension, I showed that a normal variation of spring will enable the oscillations of the system to be maintained, when the frequency of these oscillations is sufficiently nearly equal to $\frac{1}{2}$ of, or $\frac{2}{3}$ times, or $\frac{3}{4}$ times, or $\frac{4}{5}$ times, etc., the frequency of variation of the spring, these ratios forming an ascending series. By experiments on the vibrations of a body about a position of equilibrium in a periodic field of force (to be described below), I have shown that the frequency of the oscillations maintained may be equal to, or half of, or one-third, or one-fourth etc. of the frequency of the field, in other words, it may be any one of a descending series of sub-multiples of the frequency of the field. It appears, in fact, that we have here an

entirely new class of resonance-vibrations. It will be noticed that if the two series referred to above are both written in the same order of descending magnitudes of frequency, thus,

$$\frac{6}{2}, \frac{5}{2}, \frac{4}{2}, \frac{3}{2}, \frac{2}{2}, \frac{1}{2},$$

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$$

the last two terms of the first series, and the first two of the second series coincide, and these two are to some extent typical of the rest. For, as I have shown in section IV of Bulletin No. 6, the 1st, the 3rd, the 5th and the odd types generally in the first series bear a family resemblance to each other, giving symmetrical vibration curves. The 2nd, the 4th and the other even types similarly resemble each other in giving markedly *asymmetrical* vibration curves. Since the first term in the ascending series is the 2nd in the descending series, we may expect that the 2nd, 4th, 6th etc. in the latter would give analogous types of motion, and that similarly the 1st, 3rd, 5th etc. would show resemblances amongst each other. These points will be dealt with more fully as we proceed.

The vibrations studied which form the subject of this section were those of the armature-wheel of a synchronous motor of the attracted-iron type, about a position of equilibrium in the magnetic field produced by an intermittent current circulating in the coils of an electromagnet. The phonic wheel or synchronous motor devised by La Cour and Lord Rayleigh is, as is well known, of great service in acoustical investigations. In my own work on vibrations and their maintenance, it has been of considerable assistance, *vide* Sections I, II, and IV to VI of Bulletin No. 6. Apart however from the various uses of the instrument in different branches of Physics and in Applied Electricity, it possesses much intrinsic interest of its own as an excellent illustration of the dynamics of a system moving in a periodic field of force, and the present paper deals almost entirely with experiments carried out by its aid and with its applications to the study of vibrations.

The instrument used by me was supplied by Messrs. Pye & Co. of Cambridge and has given entire satisfaction. The motor consists of a wheel of soft iron mounted on an axis with ball-bearings between the two poles of an electromagnet placed diametrically with respect to it. The wheel has thirty teeth, and

when a direct current is passed through the electromagnet, sets itself rigidly at rest with a pair of teeth at the ends of a diameter opposite the two poles of the electromagnet. The equilibrium under such conditions is of course thoroughly stable, and, in fact, the wheel possesses a fairly high frequency of free angular oscillation for displacements from this position of rest, and any motion set up by such displacement rapidly dies out, apparently on account of Foucault currents induced in the iron by the motion. This, in general, is also true when an intermittent current supplied by a fork-interruptor is used to excite the electromagnet, except however in certain cases, when it is observed that the equilibrium becomes unstable of its own accord and the wheel settles down into a state of steady vigorous vibration about the line of equilibrium: or that an oscillation of sufficient amplitude once started maintains itself for an indefinitely long period.

An optical method can be conveniently used to study the frequency and the phase of the oscillations of the armature wheel maintained in the manner described above. A narrow pencil of light is used, which first suffers reflexion at the surface of a small mirror attached normally to one of the prongs of the fork-interruptor furnishing the intermittent current, and then falls upon a second similar mirror attached to the axle of the armature-wheel parallel to its axis of rotation. The apparatus is so arranged that the angular deflexions produced by the oscillations of the fork and the wheel are at right angles to each other, and the pencil of light which falls upon a distant screen, or which is focussed on the ground-glass of the photographic camera, is seen to describe a Lissajous figure from which the frequency, and the phase-relations between the oscillations of the fork-interruptor and of the armature-wheel, can be readily ascertained.

Figs. 1 to 6, Plate VII, reproduce photographs of the curves secured in this manner. The vertical motion is that due to the fork and the horizontal motion that of the armature-wheel. Of these, figs. 2 to 5 were obtained using a fork-interruptor of frequency about 24 per sec., and fig. 1 with a fork of somewhat smaller frequency. It will be observed that the periods of the vibration of the armature-wheel as shown in figs. 1 to 6 are respectively equal to, twice, thrice, four times, five times and six times

the period of the fork: in other words the frequency is equal to or $\frac{1}{2}$ of or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$ or $\frac{1}{6}$ that of the fork.

In taking the photographs, the motor-wheel was relieved of the large stroboscopic disk that is usually mounted upon it, and in working down the series the adjustment of frequency is secured by suitably loading the wheel. The *fine* adjustment for resonance is effected by altering the current passing through the interruptor with the aid of a rheostat, and if necessary by regulating the contact-maker on the fork. Any oscillation of the wheel, when started, dies away except in the cases referred to above, in other words no frequencies intermediate between those of the series are maintained. In obtaining fig 1 in which the oscillations of the wheel and the fork are shown in unison, it is generally found necessary to increase the 'spring' of the wheel by passing a steady direct current through the electromagnet of the motor from a cell connected in parallel, in addition to the intermittent current flowing in the same direction from the interruptor circuit.

It will be seen that the Lissajous figures for the 1st, 3rd and 5th cases are distinctly asymmetrical in character, the 3rd (shown in fig. 3) being markedly so. The 2nd, 4th and 6th types are quite symmetrical. This, it will be remembered, was what was anticipated above, and in fact the 1st, 3rd and 5th types differ rather markedly in their behaviour from the 2nd, 4th and 6th types. These latter are maintained with the greatest ease, while the former, particularly the 5th, are not altogether so readily maintained. In fact it is found advantageous, in order to maintain the 5th type steadily, to load the wheel somewhat unsymmetrically and to put it a little out of level, in order to allow the oscillations to take place about an axis slightly displaced from the line joining the poles of the electromagnet.

It will be noticed also that the lower frequencies of vibration have much larger amplitudes. This, I would attribute principally to the greatly reduced damping at the lower frequencies owing to the slower motion, the larger masses and the weaker magnetic fields employed.

We are now in a position to consider the mathematical theory of this class of maintained vibrations. To test the correctness of my theoretical work, I have prepared a series of photographs of the

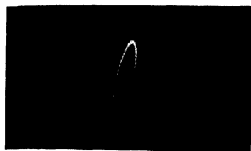


Fig. 1.

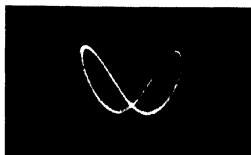


Fig. 2.



Fig. 3.



Fig. 4.



Fig. 5.



Fig. 6.

TO ILLUSTRATE THE MAINTAINANCE OF VIBRATIONS BY A PERIODIC FIELD OF FORCE, THE MAINTAINED VIBRATION IS HORIZONTAL AND THE VERTICAL MOTION REPRESENTS THE PERIODICITY OF THE FIELD.

simultaneous vibration-curves of the fork and of the armature-wheel, which are reproduced as figs. 1 to 3, Plate VIII, and figs. 1 to 3, Plate IX. These curves were obtained by the usual method of recording the vibrations optically on a moving photographic plate, it being so arranged that the directions of movement of the two representative spots of light on the plate lie in the same straight line. The upper curve in each case shows the maintained vibration of the armature-wheel. The lower represents that of the fork-interruptor. The frequency of the former, it will be seen, is $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$ or $\frac{1}{6} \times$ that of the latter. The precise features of the vibration-curve noticed in each case will be referred to below, in connection with the mathematical discussion.

The equation of motion of a system with one degree of freedom moving in a periodic field of force, and subject also to the usually assumed type of viscous resistance, may be written in the following form,

$$\ddot{U} + k\dot{U} + 2af(t)F(U) = 0 \quad (1)$$

where $F(U)$ gives the distribution of the field, $f(t)$ its variability with respect to time, and $2a$ is a constant. If we are dealing with oscillations about a position that would be one of stable equilibrium if the field were constant, $F(U)$ may as an approximation be put equal to U . We then have

$$\ddot{U} + k\dot{U} + 2af(t)U = 0 \quad (2)$$

In the experiments described above, the periodicity of $f(t)$ is the same as that of the intermittence of the exciting current. If an alternating current had been used, the frequency of $f(t)$ would have been double that of the alternations. In any case we may write

$$af(t) = a_1 \sin nt + a_2 \sin 2nt + a_3 \sin 3nt + \text{etc.} \\ + b_0 + b_1 \cos nt + b_2 \cos 2nt + b_3 \cos 3nt + \text{etc.} \quad (3)$$

Since U is shown to be periodic by experiment, we may write

$$U = A_1 \sin pt + A_2 \sin 2pt + A_3 \sin 3pt + \text{etc.} \\ + B_0 + B_1 \cos pt + B_2 \cos 2pt + B_3 \cos 3pt + \text{etc.} \quad (4)$$

As a typical example of the even types of maintenance, we may take the cases in which $n = 4p$. We have

$$af(t) = a_1 \sin 4pt + a_2 \sin 8pt + a_3 \sin 12pt + \text{etc.} \\ + b_0 + b_1 \cos 4pt + b_2 \cos 8pt + b_3 \cos 12pt + \text{etc.} \quad (5)$$

In this case, and also in the case of the second, sixth and in fact in all the *even* types of maintenance, we find that the quantities A_2, A_4, A_6 , etc., and B_0, B_2, B_4 , etc., do not enter into the equations containing A_1 and B_1 . We therefore write them all equal to zero. The significance of this is that with the *even* types of vibration maintained by a periodic field of force, the *even harmonics are all absent from the maintained motion*. This result is fully verified by a reference to the vibration-curves of the 2nd, 4th and 6th types shown in fig. 2, Plate VIII, and figs. 1 and 3, Plate IX. It will be seen that the vibratory motion of the armature-wheel has that type of symmetry so familiar in alternating current curves, in which all the even harmonics are absent. In other words, the image of one-half of the curve above the zero axis as seen by reflexion in a mirror placed parallel to this axis, is exactly similar to the other half below it.

Substituting now the odd terms alone left on the right-hand side of (4), for U in equation (2), we have the following series of equations:—

$$\begin{aligned} -(b_0 - p^2)A_1 + kpB_1 &= -b_1A_3 + a_1B_3 + b_1A_5 - a_1B_5 - \text{etc.} + \text{etc.} \\ -(b_0 - p^2)B_1 - kpA_1 &= a_1A_3 + b_1B_3 + a_1A_5 + b_1B_5 + \text{etc.} + \text{etc.} \\ -(b_0 - 9p^2)A_3 + 3kpB_3 &= -b_1A_1 + a_1B_1 - b_3A_5 + a_3B_5 + \text{etc.} - \text{etc.} \\ -(b_0 - 9p^2)B_3 - 3kpA_3 &= a_1A_1 + b_1B_1 + a_3A_5 + b_3B_5 + \text{etc.} + \text{etc.} \end{aligned} \quad (6)$$

and so on.

Evidently, the possibility of this being a consistent set of convergent equations depends upon the suitability of the values assigned to the constants k, p, b_0, a_1, b_1 , etc.

It is not possible here to enter into a complete discussion of the solution of these equations. One point is however noteworthy. From the first two of the set of equations given above, it will be seen that such of the *harmonics* in the steady motion of the system as are present serve as the vehicles for the supply of the energy requisite for the maintenance of the fundamental part of the motion. Paradoxically enough, the frequency of none of these harmonics is the same as that of the field.

We now proceed to consider the odd types of vibration, i.e. the 1st, the 3rd, etc. Taking the 3rd as a typical case, we put $n = 3p$ and get

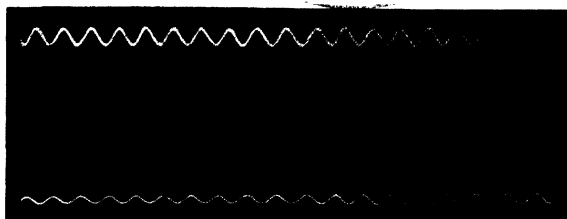


Fig. 1.

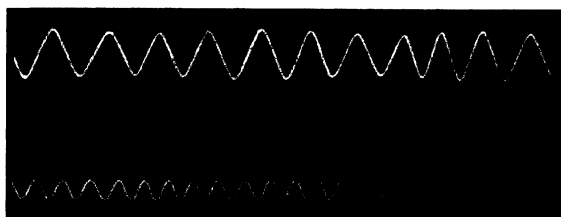


Fig. 2



Fig. 3.

VIBRATION-CURVES OF OSCILLATIONS MAINTAINED BY A PERIODIC IMPULSIVE FIELD OF FORCE,
THE UPPER CURVE IN EACH CASE REPRESENTS THE MAINTAINED MOTION.

$$af(t) = a_1 \sin 3pt + a_3 \sin 6pt + a_9 \sin 9pt + \text{etc.} \\ + b_0 + b_1 \cos 3pt + b_3 \cos 6pt + b_9 \cos 9pt + \text{etc.} \quad (7)$$

Substituting (4) and (7) in equation (2) and equating the coefficients of sine and cosine terms of various periodicities to zero, we find that the quantities A_3, A_6, A_9 , etc., and B_0, B_3, B_6, B_9 , etc., do not enter into the equations containing A_1 and B_1 . We therefore write them all equal to zero. The significance of this is that the maintained motion contains no harmonics, the frequency of which is the same as, or any multiple of the frequency of the periodic field of force. This remarkable result is verified by a reference to fig. 3, Plate VII, from which it is seen, that the vibration curve is roughly similar to that of the motion of a trisection point of a string bowed near the end, the 3rd component, the 6th, the 9th, etc., being absent at the point of observation.

We then obtain the following set of relations by substitution :

$$\begin{aligned} -(b_0 - p^2)A_1 + kpB_1 &= -b_1A_3 + a_1B_3 + b_1A_9 - a_1B_9 - \text{etc.} \\ -(b_0 - p^2)B_1 - kpA_1 &= a_1A_3 + b_1B_3 + a_1A_9 + b_1B_9 + \text{etc.} \\ -(b_0 - 4p^2)A_3 + 2kpB_3 &= -b_1A_1 + a_1B_1 + b_1A_9 - a_1B_9 + \text{etc.} \\ -(b_0 - 4p^2)B_3 - 2kpA_3 &= a_1A_1 + b_1B_1 + a_1A_9 + b_1B_9 + \text{etc.} \end{aligned}$$

and so on.

(8)

It must be remembered that these relations are all only approximate, as $F(U)$ in general contains powers of U higher than the first which we have neglected, and which no doubt must be taken into account in framing a more complete theory. The general remarks made above with reference to equation (6) apply here also.

The exact character of the vibratory motion maintained by the periodic field of force in any case, depends upon the form of the functions $F(U)$ and $f(t)$ which determine respectively the disposition of the field and its variability with respect to time. One very simple and important form of $f(t)$ is that in which the field is of an impulsive character, in other words is of great strength for a very short interval of time comprised in its period of variation, and during the rest of the period is zero or nearly zero. Such a type of variation is not merely a mathematical possibility. In actual experiment, when a fork-interruptor is used to render the current passing through the electromagnet intermittent, the magnetization of the latter subsists only during the small fraction of

the period during which the current flows and at other times is practically zero. When the current is flowing, the acceleration is considerable: at other times, the acceleration is nearly zero, and the velocity practically constant. These features are distinctly shown in all the vibration-curves (except those of the first type) reproduced in Plates VIII and IX, the sudden bends in the curves corresponding roughly to the extreme outward swings of the fork, i.e. to the instants when the magnetizing current was a maximum. It seems possible that a simpler mathematical treatment than that given above might be sufficient to discuss the phenomena of the maintenance of vibrations by a periodic field of force when the periodicity of the field is of the 'impulsive' type, in other words when the dynamical system is subject to periodic impulsive 'springs,' one, two, three or more of which occur at regular intervals during each complete period of the vibration of the system.

These experiments on vibrations maintained by a periodic field of force are very well suited for lecture demonstration, as the Lissajous figures obtained by the method described above can be projected on the screen on a large scale, and form a most convincing demonstration of the fact that the frequency of the maintained motion is an exact sub-multiple of the frequency of the exciting current.

ON SYNCHRONOUS ROTATION UNDER SIMPLE EXCITATION.

It is well known that with an intermittent current passing through its electromagnet, the synchronous motor can maintain itself in 'uniform' rotation, when for every period of the current, one tooth in the armature-wheel passes each pole of the electromagnet. In other words, the number of teeth passing per second is the same as the frequency of the intermittent current. From a dynamical point of view it is of interest, therefore, to investigate whether the motor could run itself successfully at any speeds other than the 'synchronous' speed. Some preliminary trials with the motor unassisted by any independent driving proved very encouraging. The phonic wheel I have is mounted on ball-bearings, and runs very lightly when the large stroboscopic disc usually kept fixed upon it is taken off, and there is no current passing through the motor. When a continuous or intermittent current is flowing

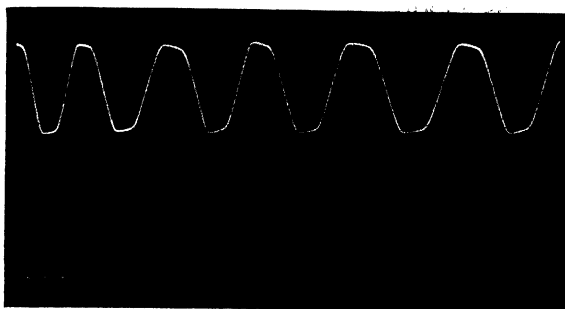


Fig. 1.

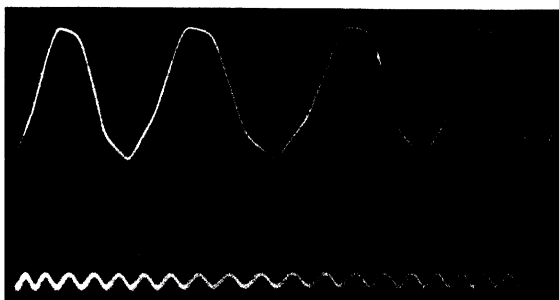


Fig. 2.

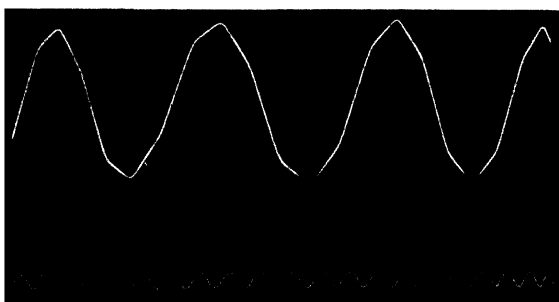


Fig. 3.

VIBRATION-CURVES OF OSCILLATIONS MAINTAINED BY A PERIODIC IMPULSIVE FIELD OF FORCE.
THE UPPER CURVE IN EACH CASE REPRESENTS THE MAINTAINED MOTION.

through the motor, the latter does not however run very lightly, being subject to very large electromagnetic damping apparently due to Foucault currents in the iron. In the preliminary trials, however, I found that, using the intermittent unidirectional current from an interruptor-fork of frequency 60, the motor could run successfully of itself at *half* the synchronous speed, i.e. with 30 teeth passing per second. It of course ran very well at the usual synchronous speed, i.e. with 60 teeth passing per second. By increasing the speed, it was found that the motor could also run well of itself at *double* the synchronous speed, i.e. with 120 teeth passing per second. Using an interruptor-fork of low frequency (23.5 per second) the motor, it was found, could also run of itself at *triple* the synchronous speed. No certain indication was however obtained of the intermediate speeds, i.e. $1\frac{1}{2}$ and $2\frac{1}{2}$ times respectively the synchronous speed.

To test these points, therefore, independent driving was provided. This was very satisfactorily obtained by fixing a small vertical water-wheel to the end of the axis of the motor and directing a jet of water against it. The water-wheel was boxed in to prevent any splashing of water on the observer. By regulating the tap leading up to the jet, the velocity of the latter could be adjusted. The speed of the phonic wheel was ascertained by an optical method, i.e., by observing the rim of the wheel as seen reflected in a mirror attached to the prong of the interruptor-fork. When the motor 'bites,' the pattern seen becomes stationary and remains so for long intervals of time or even indefinitely, and the speed of the wheel can be inferred at once from the nature of the pattern seen.

It was found in these trials that the motor could 'bite' and run at the following speeds. (Frequency of interruptor 60 per sec.).

(a) $\frac{1}{2}$ the synchronous speed: stationary pattern of rim of moving wheel seen as a single sine-curve: wave-length $\frac{1}{2}$ the interval between teeth. Number of teeth passing electromagnet per second=30.

(b) Synchronous speed: stationary pattern of rim of moving wheel seen as a sine-curve, wave-length=interval between teeth. Number of teeth passing electromagnet per second=60.

- (c) $1\frac{1}{2}$ times the synchronous speed : stationary pattern of rim of wheel seen as *three* interlacing waves. Number of teeth passing electromagnet per second=90.
- (d) 2 times the synchronous speed : stationary pattern seen as *two* interlacing curves. Number of teeth passing per second=120.
- (e) $2\frac{1}{2}$ times the synchronous speed : this was only obtained with difficulty. Number of teeth passing per second=150.
- (f) 3 times the synchronous speed : stationary pattern seen as *three* interlacing curves. Very satisfactory running. Number of teeth per second=180.
- (g) 4 times the synchronous speed : stationary pattern seen as 4 interlacing curves. Number of teeth per second=240.
- (h) 5 times the synchronous speed : stationary pattern seen as 5 interlacing curves. Number of teeth per second=300.

The outstanding fact of observation is that while speeds which are equal to the 'synchronous' speed or any integral multiple of it are readily maintained, only the first two or three members of the other series (i.e. having ratios $\frac{1}{2}$, $1\frac{1}{2}$ etc. to the synchronous speed) can be obtained and the 'grip' of the wheel by the periodic magnetic forces, i.e. the stability of the motion, is hardly so great as in the integral series. This fact may be explained in the following general manner.

We may assume, to begin with, that the independent driving is less powerful than that required to overcome resistances, so that the wheel is a little *behind* the correct phases. In the case of the integral series, one or two or more teeth pass for every intermittence of the current, the wheel being in the same relative position, whatever this may be, to the electromagnet, at each phase of maximum magnetization of the latter. This is not, however, the case with the fractional speeds. It is only at every *alternate* phase of maximum magnetization that the wheel assumes the same position (whatever this may be) relative to the electromagnet. At the intermediate phases, it is displaced through a distance approximately equal to half the interval between the teeth. Whereas

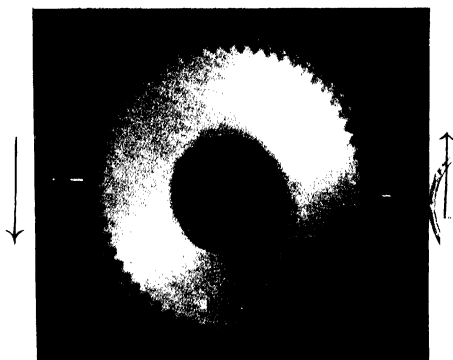


Fig. 1.
SYNCHRONOUS MOTOR, ROTATING AT HALF-SPEED (ANTI-CLOCKWISE)
PHOTOGRAPHED UNDER PERIODIC ILLUMINATION.

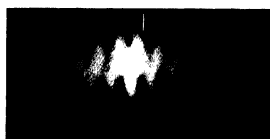


Fig. 2.



Fig. 3.

EDGE OF WHEEL PHOTOGRAPHED THROUGH OSCILLATING LENS.
DIRECTION OF ROTATION INDICATED BY ARROW.

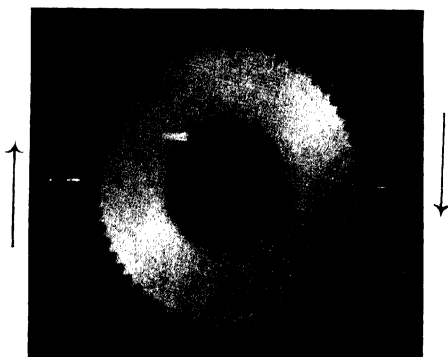


Fig. 4.
SAME AS FIG. 1. WITH DIRECTION OF ROTATION REVERSED.

with the integral series, *every* phase of maximum magnetization *assists* the rotation, in the fractional series the wheel is alternately assisted and retarded by the successive phases of maximum magnetization, and it is the *net* effect of assistance that we perceive, this being of course comparatively small.

The explanation given above may be made clearer by reference to Plates X and XI, in which are reproduced photographs of the armature-wheel in rotation taken under special arrangements for illumination. Figs 1 and 4, Plate X, show the motor in rotation at half the synchronous speed, photographed under intermittent illumination having the same frequency as the fork-interruptor. Fig. 1 refers to the anti-clockwise rotation, and fig. 4 to clockwise rotation. Figs 1 and 4, Plate XI, refer similarly to the cases of rotation at the normal synchronous speed in the anti-clockwise and clockwise directions respectively. In these four photographs the position of both the poles of the electromagnet is indicated, and the displacement of the position of the armature-wheel with respect to the poles on reversing the direction of rotation can readily be made out by comparison. From figs. 1 and 4, Plate X, it will be seen that in the case of the half-speed rotation the attraction of the poles on the armature-wheel would tend alternately to encourage and retard the motion, the difference only being the surplus available to balance the loss of energy due to frictional forces.

In taking these photographs it was arranged that the axis of the motor should point towards the camera, and the normal illumination of the wheel was secured by an arc lamp and a silvered mirror placed at an angle of 45° between the motor and camera. The removal of a small circular patch of the silvering from the back of the glass opposite the lens of the camera, and the interposition between these two of the fork-interruptor with light overlapping plates fixed on its prongs periodically to open out a passage for the light to enter the camera, completed the arrangements. It was found that rotation at double the synchronous speed could be similarly photographed: the definition was however not so good on account of the greater velocity of motion. It was found that the arrangement described above was far more satisfactory than the employment of intermittent illumination

from the spark of an induction coil worked with a fork-interruptor.

As the synchronous, half-synchronous, and double-synchronous speeds can all be readily maintained without independent driving, they can be very effectively exhibited as lecture experiments by lantern projection in the following way. The synchronous motor (which is quite small and light when the stroboscopic disk is removed) is placed on the horizontal stage of the lantern and the rim of the wheel is focussed on the screen. In front of the projection prism, where the image of the source of light is formed, is placed the fork-interruptor with the necessary device for intermittent illumination fitted to its prongs. When these are set into vibration and the synchronous motor is set in rotation, the "pattern" corresponding to the maintained speed becomes visible on the screen, and the effect of reversing the direction of rotation can also be demonstrated.

Figs. 2 and 3 in Plates X and XI are photographs of the edge of the phonic wheel revolving at the half-synchronous and the synchronous speeds, taken through a lens which was fixed to one of the prongs of the interruptor-fork, and which therefore caused the image of the wheel to vibrate on the plate in a direction at right angles to the plane of rotation. In these photographs also, the displacement of the position caused by reversing the direction of rotation is clearly shown.

We now proceed to discuss the mathematical theory of the maintenance of uniform rotation in each of these cases. The first step is obviously to show that with the assumed velocity of rotation, the attractive forces acting on the disc communicate sufficient energy to it to balance the loss due to frictional forces. Taking the line joining the poles as the axis of x , the position of the wheel at any instant may be defined by the angle θ which a diameter of the wheel passing through a given pair of teeth makes with the axis of reference. If n is the number of teeth in the wheel, the couple acting on the latter for any given field strength at the poles is obviously a periodic function of $n\theta$ which vanishes when $\theta = \frac{2\pi r}{n}$,

and also when $\theta = \frac{2\pi(r + \frac{1}{2})}{n}$, where r is any integer.

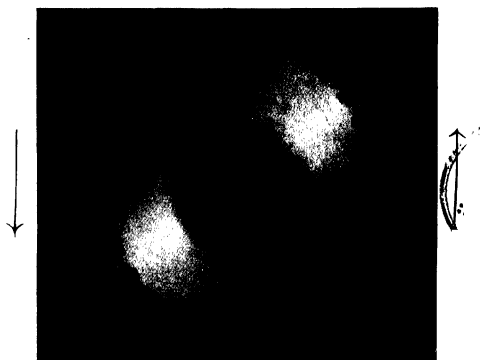


Fig. 1.
SYNCHRONOUS MOTOR, ROTATING AT NORMAL SPEED (ANTI-CLOCKWISE)
PHOTOGRAPHED UNDER INTERMITTENT ILLUMINATION.



Fig. 2.
EDGE OF WHEEL PHOTOGRAPHED THROUGH OSCILLATING LENS.
DIRECTION OF ROTATION INDICATED BY ARROW.

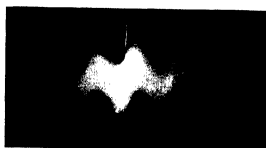


Fig. 3.

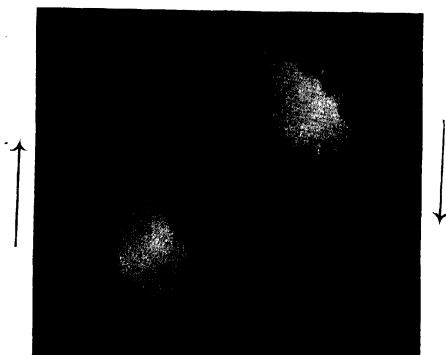


Fig. 4.

We therefore write

$$\begin{aligned}\text{Couple} &= \text{Field strength} \times [a_1 \sin n\theta + a_2 \sin 2n\theta + a_3 \sin 3n\theta + \text{etc.}] \\ &= \text{Field strength} \times f(n\theta) \text{ say,}\end{aligned}$$

where the terms a_1, a_2, a_3 , etc. rapidly diminish in amplitude. It will be seen that the cosine terms are absent. Since the field strength is periodic, we may write the expression for the couple acting on the wheel thus

$$\begin{aligned}\text{Couple} &= Lf(n\theta)[b_1 \sin(pt + \epsilon_1) + b_2 \sin(2pt + \epsilon_2) + \text{etc.}] \\ &= Lf(n\theta)F(t), \text{ say.}\end{aligned}$$

The work done by the couple in any number of revolutions

$$= \int Lf(n\theta) F(t) dt.$$

It is obvious that this integral after any number of complete revolutions is zero, except in any of the following cases, when it has a finite value proportional to and increasing with t ; i.e., when

$$n\theta = pt \text{ or } 2pt \text{ or } 3pt \text{ or } 4pt \text{ and so on}$$

or when

$$2n\theta = pt \text{ or } 2pt \text{ or } 3pt \text{ or } 4pt \text{ and so on}$$

or when

$$3n\theta = pt \text{ or } 2pt \text{ and so on.}$$

It is therefore a necessary but not, of course, always a sufficient condition for uniform rotation to be possible that one or more of the above relations should be satisfied. The first series corresponds to the synchronous speed and multiples of the synchronous speed. These have been observed experimentally by me up to the fifth at least. The second series includes the above and also the half-synchronous speed and odd multiples of the same. These latter have also been observed by me up to the fifth odd multiple. Since a_2 is much smaller than a_1 , the relative feebleness of the maintenance of the half-speeds observed in experiment will readily be understood.

The third series has not so far been noticed in experiment. It is obvious that the maintaining forces in it should be excessively feeble compared with the first or the second. Perhaps, experiments with interruptor-forks of higher frequencies and independent driving of the motor may succeed in showing the existence of controlled rotation-speeds at these ratios.

Fig. 1, Plate XII, shows a record of the rotation of the motor maintained at double the synchronous speed, side by side with the vibration-curve of the fork-interruptor which supplied the intermittent current necessary. The record of the rotation of the motor was secured by the use of a moving photographic plate, and of an illuminated slit, the light from the latter being cut off periodically by the teeth of the armature-wheel, as it revolved. An illuminated pin-hole in a small piece of metal foil attached to a prong of the fork was simultaneously photographed on the plate. It will be seen that the number of teeth of the armature-wheel that passed in any given time is double the number of vibrations of the fork recorded in the same time.

COMBINATIONAL ROTATION-SPEEDS UNDER DOUBLE EXCITATION.

When the electromagnet of the synchronous motor is excited simultaneously by the intermittent currents from two separate interruptor-forks having different frequencies, maintenance of uniform rotation is possible not only at the various speeds related to the synchronous speeds due to either of the intermittent currents acting by itself, but also at speeds related jointly to the frequencies of the two currents.

The preliminary experiments on this point were made without the assistance of any independent driving of the motor and it found at once that differential rotation of the motor was easily maintained, the number of teeth passing per second being equal to the difference of the frequencies of the two interruptor-forks. This result is shown in fig. 2, Plate XII, the motion of the wheel and of the two forks being recorded in the same way as that described in a preceding paragraph. In this case it is seen that the teeth of the rotating armature-wheel have been clearly reproduced on the moving photographic plate.

When the 'differentially' revolving wheel was examined by reflexion in mirrors attached to the prongs of the two interruptor-forks, it was found that the patterns seen in neither of them was stationary. They were found to be moving steadily forward or backward with a definite speed, with occasional slight to and fro oscillations superposed thereon. This continuous rotation of the



Fig. 1. DOUBLE - SYNCHRONOUS ROTATION OF PHONIC WHEEL.

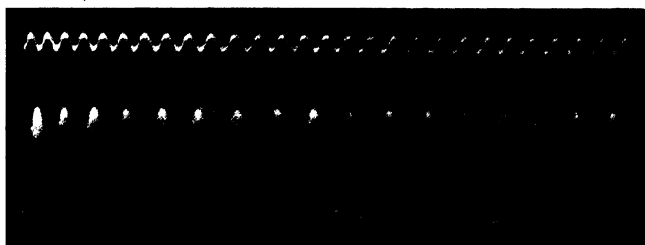


Fig. 2. DIFFERENTIAL ROTATION OF PHONIC WHEEL.



Fig. 3.



Fig. 4. STROBOSCOPIC PICTURES OF VIBRATING STRING.
(SEE ALSO PLATES XIII & XIV.)

patterns seen was obviously due to the fact that the frequencies of of the forks and their difference did not bear any simple arithmetical ratios to each other, and it enabled a rotation-speed maintained by joint action to be distinguished by mere inspection from one maintained by either of the two currents separately.

Using this optical method, and assisting the rotation of the motor with independent driving by a water-motor, various other combinational speeds were found to be maintained. Of these, the most powerfully and steadily maintained was the simple summational rotation. The summationals and differentials of the second series, i.e., those in which the half-frequencies of the fork enter, were also noticed. The rotation-speeds were determined by actual counting and a stop-watch.

The mathematical theory of these combinational speeds is very similar to that given for the case of excitation by one periodic current. For, the field strength in this case is also a periodic function of the time, and the function $F(t)$ which expresses its value at any instant may be expanded in the following form

$$F(t) = a \sum \sum b \sin[(r\phi_1 \pm s\phi_2)t + E],$$

where $\phi_1/2\pi$ and $\phi_2/2\pi$ are the frequencies of the two interruptors, and r, s are any two positive integers. Using the same notation as before, we find that in any complete number of revolutions, a finite amount of energy proportional to the time is communicated to the wheel, only in any one out of the following sets of cases:

$$n\theta = (r\phi_1 \pm s\phi_2)t$$

or

$$2n\theta = (r\phi_1 \pm s\phi_2)t$$

or

$$3n\theta = (r\phi_1 \pm s\phi_2)t$$

and so on.

The cases actually observed in which rotation is maintained fall within the first two of the sets given above.

SOME APPLICATIONS OF THE SYNCHRONOUS MOTOR.

In acoustical observations, particularly where regularly intermittent illumination is desired, the synchronous motor is of great use. This is specially the case, when it is desired that the vibra-

tions under observation should be seen under control, simultaneously or successively in various phases of motion. An excellent example where this requirement has to be met is in the study of the small motion at the nodes of a vibrating string. A series of 13 photographs of this small motion secured with a stroboscopic disc mounted on a synchronous motor was published as Plate II of Bulletin No. 6. Another very interesting application of a stroboscopic disc with radial slits mounted on a synchronous motor is for the study of the oscillations of a stretched string under a variable tension. In this case, as I have shown, the vibrations of the string can be maintained when the frequency of its free oscillations is half of, or equal to, or one and half times, etc., etc., the frequency of the fork which varies its tension. This fork is electrically self-maintained, and the intermittent current from it also feeds the synchronous motor. The number of radial slits in the stroboscopic disc is either 30 or 60, i.e., equal to, or double the number of teeth on the armature-wheel. In making the observations, the stroboscopic disc is held vertically and the string is set horizontal and parallel to the disc is viewed through the top row of slits, i.e. those which are vertical and move in a direction parallel to the string as the disc revolves. It is advantageous to have the whole length of the string brilliantly illuminated and to let as little stray light as possible fall upon the reverse of the disc at some distance from which the observer takes his stand. A brilliant view is then obtained. Under these circumstances we see the string in successive cycles of phase along its length, and the peculiar character of the maintained motion in these cases is brought out in a very remarkable way. *The string is seen in the form of a stationary vibration curve*, which would be identical with those published in Bulletin No. 6, but for the fact that the amplitude of motion is not the same at all points of the string, being a maximum at the ventral segments and zero at the nodes.

Another point calls for remark. Using a fork with a frequency of 60 per second, the *free* oscillations of the string have a frequency of 30 in the case of the 1st type, 60 in the case of the 2nd, 90 with the 3rd, 120 with the 4th, 150 with the 5th, and so on. With the disc having 30 slits on it we get 60 views per second of any one point on the string, and with the even types of motion,



Fig. 1. FIRST TYPE : 30-SLOT DISK.



Fig. 2. FIRST TYPE : 60-SLOT DISK.



Fig. 3. SECOND TYPE : 30-SLOT DISK.

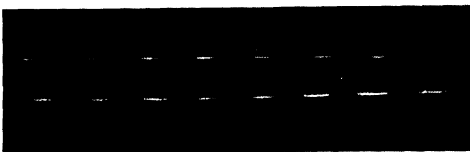


Fig. 4. SECOND TYPE : 60-SLOT DISK.

VIBRATIONS OF A STRETCHED STRING MAINTAINED BY A VARIABLE TENSION, OBSERVED THROUGH A STROBOSCOPIC DISK MOUNTED ON A SYNCHRONOUS MOTOR.

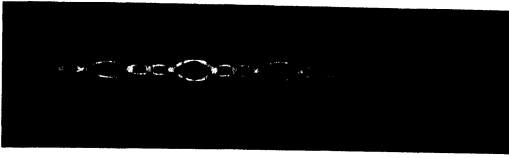


Fig. 1. THIRD TYPE : 30-SLOT DISK.

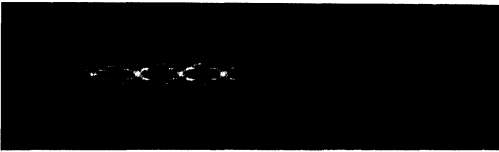


Fig. 2. THIRD TYPE : 60-SLOT DISK.



Fig. 3. FOURTH TYPE : 30-SLOT DISK.

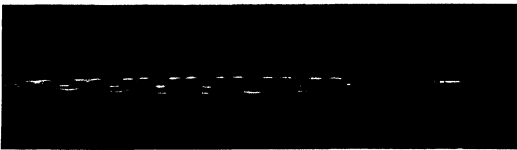


Fig. 4. FOURTH TYPE : 60-SLOT DISK.

VIBRATIONS OF A STRETCHED STRING MAINTAINED BY A VARIABLE TENSION, OBSERVED
THROUGH A STROBOSCOPIC DISK MOUNTED ON A SYNCHRONOUS MOTOR.

i.e. the 2nd, 4th, etc. the 'vibration-curve' seen through the stroboscopic disc appears single. With the odd types, i.e. the 1st, 3rd, 5th, etc., *two* vibration-curves are seen, one of which is as nearly as can be seen the mirror-image of the other, intersecting it at points which lie or should lie upon the equilibrium position of the string. The reason why with the odd types we see the vibration-curve double is fairly clear and furnishes an excellent illustration of the principles of stroboscopic observation. The double pattern brings home to the eye in a vivid and convincing manner the fact that under the action of the variable spring the 'amplitude' and 'period' of the motion periodically increase and decrease after the manner of 'beats.'

An interesting variation on the experiment is made by using the disc with 60 slits. We then get 120 views per second and with the even types we get the vibration-curves double, but one of the curves is not the mirror image of the other, the motion not being symmetrical. On the other hand, with the odd types we see the vibration-curves in quadruple pattern.

Plates XIII and XIV, and figs. 3 and 4, Plate XII, reproduce photographs secured with the apparatus described above.

Figs. 1 and 2, Plate XIII, represent the 1st type of maintenance observed through the 30-slot and the 60-slot discs respectively.

Fig. 3, Plate XIII, represents the 2nd type seen through the 30-slot disk and fig. 3, Plate XII, and fig. 4, Plate XIII, represent this type as seen through the 60-slot disc.

Figs. 1 and 2, Plate XIV, represent the 3rd type as seen through the two discs. Figs. 3 and 4, Plate XIV, similarly show the 4th type. Fig. 4, Plate XII, shows a compound type of vibration maintained by a simple harmonic variation of tension as seen through the 30-slot disc. It will be seen that in all these photographs, there is a slight degree of distortion towards the end of the string, due to the fact that the discs are relatively of small diameter and that the ends of the string are seen through the slits in a position of the latter in which they are inclined and move in a direction inclined to the string.

The arrangement described above may also be applied to the study of the motion of a bowed string. For this purpose, the string

should be carefully tuned to be in unison with the fork-interruptor running the synchronous motor. With this arrangement, it is possible at one glance to recognize the modes of vibration at all points of the bowed string simultaneously. The definition of the curve seen is fairly good in the case of the simpler types, but it is not easy or possible to secure satisfactory photographs of the configuration of a bowed string by this method, on account of the appreciable width of the slots on the disc

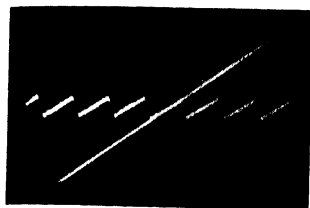


Fig. 1.

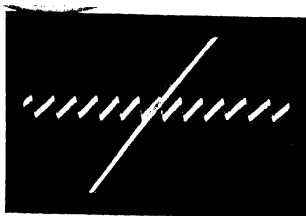


Fig. 2.

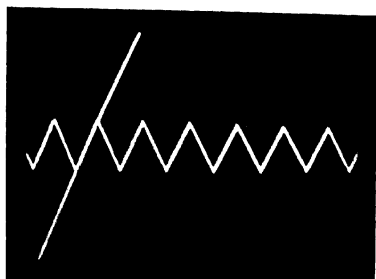


Fig. 3. SIMULTANEOUS RECORDS OF MOTION OF A VOILIN BOW AND OF THE BOWED POINT.

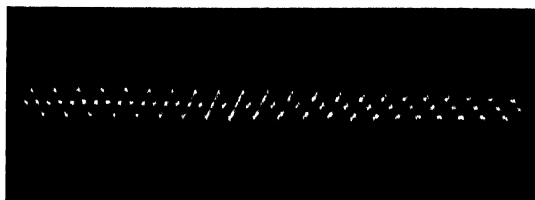


Fig. 4. RECORD FROM MOVING SLIT APPARATUS. (SEE PLATE XVII.)

PLATE XVI.



Fig 1



Fig 2



Fig 3



Fig 4



Fig 5



Fig 6



Fig 7



Fig 8



Fig 9

SOME PHOTOGRAPHS OF VIBRATION-MICROSCOPE FIGURES OF BOWED STRINGS.

III. The Dynamical Theory of the Motion of Bowed Strings.

By C. V. Raman, M.A.

(Plates XV to XVII.)

SUMMARY OF CONTENTS.

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The Projection and Photography of Vibration-Microscope Figures ..	50

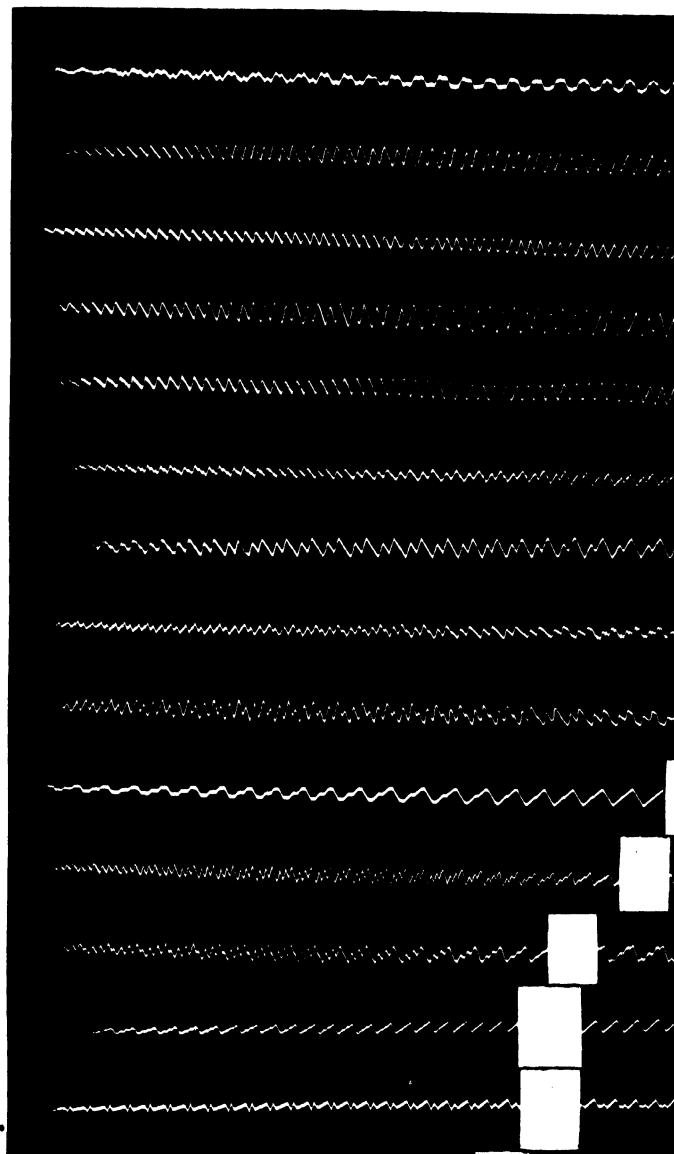
PREFATORY NOTE.

Our knowledge of the motion of bowed strings derived from the researches of Helmholtz and his successors is mainly of a kinematical character. The present investigation was prompted by a desire to attain a fuller understanding and appreciation of the distinctive *dynamical* as distinguished from the purely geometrical features of the problem. The attack can evidently be approached from two sides. One is the purely *a priori*, abstract, theoretical point of view. The other procedure is to seek out fresh experimental data on which a superstructure of dynamical theory can be built up. The paper now presented is of a preliminary character and deals principally with certain points which do not appear to have been generally recognized or emphasized. Incidentally a new kinematical method of recording the vibrations of the entire length of a bowed string in one photograph is developed and sixteen pictures obtained by this method are presented (Plate XVII). Methods of projecting vibration-microscope figures with the lantern and of photographing them are described towards the end of the paper. Some photographs of these figures are presented in Plate XVI.

THE MODUS OPERANDI OF THE VIOLIN BOW.

Obviously in any dynamical theory, the exact relation between the motion of the bow and that of the bowed point is of the very first importance. Helmholtz, referring to the particular case in which the motion of the bowed point is represented by the two-step zig-zags discussed by him, remarks (*vide* Sensations of Tone, English Translation, page 83) that the velocity in the forward motion "appears" to be equal to that of the bow. Later treatises state that the velocity of the forward motion is probably equal or about equal to that of the bow (e.g., Lord Rayleigh's Theory of Sound, Vol. I, and Barton's Text-Book of Sound, Art. 261). It is evident that an exact and convincing demonstration of this equality of velocities has not so far been given, and such a proof is obviously of importance, for, in a very large number of cases (*vide* researches of Helmholtz and those of Krigar-Menzel and Raps, Sitzungberichte of the Berlin Academy, 1891) the vibration-curve of the bowed point is exactly represented by a two-step zig-zag, the ratio of the forward and backward velocities bearing some relation to the position of the bowed point on the string.

The following method has been devised by me for obtaining simultaneous photographic records of the motion of the bow and of the bowed point in contact with it. A fairly long string is chosen for the experiment and a narrow slit cut in a sheet of metal is set across and immediately behind the string. The positive crater of a small arc placed in front of the string provides the necessary illumination. The string is bowed at a point as close as possible to the position of the illuminated slit. A pin is fixed transversely at the centre of the bow. A photographic plate contained in a dark slide is arranged to slide in grooves parallel to the string, and the movement is timed so that the shadow of the pin fixed to the bow passes across the illuminated slit when the photograph is being secured. We thus obtain a simultaneous record of the motion of the bow and of the vibrations of the bowed point on the string. When the apparatus has been correctly adjusted, we find that the record of the bow is absolutely parallel to that of the forward motion of the string. In some cases the two records become perfectly *coincident* (this, of course, is mere



chance) and the picture becomes a most convincing demonstration of the equality of velocities. Three such records are shown in figs. 1, 2 and 3, Plate XV.

Working by this method, we arrive at the generalization that in every case in which the motion of the bowed point is a two-step zig-zag, the velocity of the forward motion is *accurately* equal to that of the bow. This fact has obviously a bearing on the dynamical theory of the maintenance of the periodic motion.

It is a somewhat remarkable fact that the current treatises in describing the form of a bowed string at any instant during its motion (even in the simplest cases), overlook the effect of the friction of the bow on the configuration of the stretched string and thus convey a somewhat erroneous idea of the *modus operandi* of the bow. The first effect of the bow must evidently be to cause a deflection of the string in the direction of its movement into two lines meeting more or less sharply at the point of application, and a little consideration will show that the vibrations started and maintained by the bow may to some extent modify but cannot abolish this primary effect. Assuming, for the sake of clearness in ideas, that the region of application of the bow is confined to an extremely short length of the string, and that θ is the angle between the two lengths of string or either side of it, it follows (with a reservation) that the value of θ at any instant during the vibration is given by the equation,

$$\text{Frictional Force acting on String} = \text{Tension} \times \sin(\pi - \theta).$$

The reservation is that we must exclude the instants at which θ suffers an infinitely sudden increase or decrease followed by an infinitely sudden change in the opposite direction owing to the passage of a crest of the wave through the bowed point. On account of the (ordinarily) very short duration of these 'discontinuous' changes and the fact that the forces acting on the string are finite, no appreciable energy passes into the string at these instants. If the value of θ is taken as constant throughout the rest of the motion, we arrive at the result that there is apparently nothing to enable the vibrations of the string to be maintained in the presence of dissipative forces. In other words, we find that the expression for the motion of the string given by

Helmholtz's formulae are quite incapable of explaining the maintenance of the motion. The inference to be drawn is that the kinematical expressions for the motion are incomplete, and certain additional dynamical terms must be added to enable an accurate idea of the motion to be obtained. These dynamical terms involve a certain periodic variation of the angle θ .

From purely *à priori* considerations, we thus arrive at the following conclusions: the form of the bowed string at any instant in the simplest case considered by Helmholtz is in general that of *three* straight lines meeting at sharp angles (not two, as is generally supposed). Further, two of these straight lines always meet at the bowed point, and the angle between them suffers a small periodic variation which is distinct from the sudden increase followed by the sudden decrease caused by the passage of a 'crest' of the wave through the bowed joint. This small periodic variation is necessary to enable the maintenance of the motion to be explained and involves a modification of the usual kinematical expression for the motion at all points on the string, small no doubt, but theoretically of great importance. The most promising method of verifying the above-stated conclusions would appear to be to obtain a series of shadow photographs of the form of the string, at and near the bowed point, on a falling plate under periodic illumination secured by electric sparks of short duration, following one another at intervals of a tenth or a twelfth of the period of vibration of the string. Experiments by this method are under contemplation.

Another line of reasoning leads to conclusions precisely similar to those stated above. It is known that by bowing in a suitable manner near the end of the string, or better still near its middle point, it is possible to elicit the octave of the string without any admixture of the fundamental motion. In this case, the centre is a node, and to admit of the passage of the energy requisite for the maintenance of the vibrations of the second half of the string there must be a small motion at the node, the phase of which for each periodic component differs from that of the large motion elsewhere by quarter of an oscillation.¹ The kinematically deduced

¹ See "The Small Motion at the Nodes of a Vibrating String," Section II, Bulletin No. 6.

expressions obviously fail to account for the existence of this small motion and must therefore be modified accordingly. This small motion appears to be distinctly present and is shown in record No. 5 in Plate XVII, obtained by a method to be described below.

The preceding discussion when carried a step further enables us to deduce from purely *a priori* considerations some of the possible forms of vibration of bowed strings. It is obvious from dynamical considerations that in steady motion the velocity of the bowed point at any instant in the direction of bowing cannot normally exceed that of the bow itself. For, if it did, the direction of the frictional force would be reversed and the extra velocity would be quickly damped down. Again, as we have already seen, the maintenance of the motion depends upon a periodic variation of the frictional force acting on the string, and if, as is generally the case, the damping of the vibrations is very small, the motion approximates more and more closely to a type in which there is no damping, and no maintenance, in other words to a free oscillation subject to no periodic forces. The string possesses an infinite number of degrees of freedom, and it follows that, provided the pressure and velocity of bowing lie within certain limits, it would normally adjust itself as closely as possible to a type of motion in which the frictional force due to the bow is constant. Since this friction is a function of the relative velocity, periodic terms can be absent from it only when the forward and backward velocities of the bowed point are each of them constant, and the forward velocity is equal to that of the bow. Such a type of motion in the case of a stretched string is possible, provided the ratio of the forward and backward velocities is such that the motion, when analysed and expressed in a Fourier series, does not contain any harmonic, a node of which coincides exactly with the point of bowing. For example, if the node is that of the n th harmonic, the ratios of the forward and backward velocities may be one of the ratios $\frac{r}{n-r}$ where r is an integer less than n . The particular ratio which obtains depends of course on the pressure and velocity of bowing. These inferences are generally in accord with the results of experiment.

A NEW KINEMATICAL METHOD OF RECORDING THE ENTIRE MOTION OF A BOWED STRING.

It is well known that a bowed string is capable of many complicated types of vibration. To convey a really accurate and effective idea of the motion of the string in any of these cases, it would be necessary to observe or photograph the vibration-curves of many individual points on the string, care being taken, when each photograph is secured, that the string is bowed precisely at the same point and as nearly as possible with the same pressure and velocity. When this is done, it would be necessary also to indicate the place of bowing and the point of observation against each photograph, and if possible to arrange these in some kind of geometrical order, so that the eye can realize the configuration of the string as a whole. Now it would obviously be a great gain, if the whole business could be managed in one operation. It is well known that, after a little practice, it is much easier to maintain a nearly uniform style and place of bowing for, say half a second, than to reproduce these correctly 15 or 20 times in several successive efforts. It will also be conceded that 80 or 90 vibration-curves of successive points 'stringed' together in one photograph will convey to the eye far more of the finer effects of the gradation of the motion from point to point, than 5, 10 or even 15 separate photographs could. This result I have succeeded in securing and 16 records are presented in Plate XVII.

The experimental procedure is very simple indeed. A fine violin string is stretched on low bridges mounted on a hollow-box. The top of the box between the two bridges is open and fitted with grooves inside which a long strip of wood about 2 inches wide slides very smoothly parallel to the string. When the apparatus is not at work, the strip completely covers the opening, so that the box is light-tight. An aperture in the strip is fitted with a fine slit transverse to the string. A small brilliant arc placed at some distance in front of the box provides the source of illumination. An arrangement is made by which a length of sensitive film or bromide paper is held up just behind the sliding cover. The string is bowed steadily, and the illuminated slit with the shadow of the string across it is caused to slide with uniform velocity from

end to end of the string, and thus to record the motion of the latter on the sensitive surface.

The necessary uniform movement of the slide carrying the slit is secured by the aid of a small fly-wheel and a string wrapped round its axle. A little string is left slack at first while the fly-wheel is set into rotation, and the string tightens and moves the slide forward with uniform velocity. When the slit has passed over the whole of the string, the slide comes up against a stop and the connecting string snaps off.

The place of bowing is indicated in the photographs by the shadow of the bow crossing the string. The vibration-curves come right up to this short length on either side, and the nature of the motion at the bowed point itself is thus sufficiently indicated. It will be seen that in Plate XVII, each record consists of 90 to 100 vibration-curves or 'waves' continuously stringed together. The length of each of these waves is the distance through which the slit travelled in one complete period of oscillation of the string, and they represent with perfect fidelity the time-displacement diagrams of the points on the string at which they respectively appear. For, the length of the 'wave' is a very small fraction of the length of the string, in fact much smaller than the length of the ventral segments of all harmonics which have sensible amplitudes, i.e., say up to the 10th or 11th; and further, discontinuous changes of velocity which involve harmonics of very high orders are also reproduced in the curve as discontinuous changes in direction.

In other words, these curves are for all practical purposes precisely similar to those obtained for individual points on the string by the method of a stationary slit and moving photographic plate. Each 'wave' differs from those on either side of it by almost imperceptible gradations (see for instance a section reproduced as fig. 4, Plate XV), and thus, an accurate picture of the vibration of all points on the string is presented to the eye.

This method, of course, is suitable for the photographic record of all cases of the vibrations of strings or thin rods, provided these oscillations are maintained steadily during the interval the illuminated slit takes to pass over from end to end of the string or thin rod, or over any desired section of it. In the case of damped oscilla-

tions, e.g., those produced by striking or plucking, the method is not so effective. Even here, however, some use can be made of the method, and to illustrate this, I have included the case of a plucked string (Record No. 17).

In the case of bowed strings, when the bowing has been fairly steady, the whole picture, in addition to conveying the vibration-form for individual points on the string, also indicates the configuration of the string as a whole, i.e., shows the relative amplitudes of the motion at various sections, and also the deflexion of the string due to the friction of the bow, which I have referred to previously. Further, the curves when held nearly in a line with the eye, show very well the division of the string into alternately brighter and darker strips parallel to its length over various sections of it, which are so readily observed by the unassisted eye when a bowed string is seen under good illumination. The 'bright' strips are seen over the regions where, for a short distance, points on the string reverse their direction of motion and retrace their paths, and the 'dark' strips where the points move down with increased velocities.

THE PROJECTION AND PHOTOGRAPHY OF VIBRATION-MICROSCOPE FIGURES.

Helmholtz, as is well known, deduced the vibration-forms of bowed strings in some of the simple cases by the very beautiful, if a trifle complicated, method of the vibration-microscope. The figures seen in this instrument possess great fascination and interest to the experimenter on account, partly, of the variety and the beauty of the forms which they may assume, and also on account of the peculiar, almost stereoscopic effect produced by the slow rotation of the figures under slightly imperfect tuning. In the form devised by Helmholtz, however, the instrument can only be used by one observer at a time, and as a monocular microscope is employed, there is some discomfort in its use which renders the observation of these figures less attractive than it otherwise might be. In fact, the number of those who have seen even the simplest of these curves for themselves, i.e., elsewhere than in text-book diagrams, is probably not very large, and the interest of these figures does not appear to be very generally realized.

There are some difficulties in obtaining satisfactory projections of these figures on the screen, which I have successfully overcome. For use with an ordinary lantern, even in one in which the electric arc is employed, Helmholtz's original method is obviously quite unsuitable, as the amount of light available is altogether insufficient. The following arrangement is a very suitable one. The condenser of the lantern is covered by a plate carrying an adjustable slit with vertical jaws. A steel wire about 80 cms. long is employed as the 'string' to be bowed, and this is stretched horizontally close to the cap covering the condenser of the lantern, so as to bisect the slit. An electrically-maintained interruptor-fork of frequency, say 60, is used. This is held vertically, and upon one of the prongs of this, a small achromatic lens of short focal length, say about 7.5 cms., is fixed on with wax, the other prong is loaded to balance the masses. This lens forms an enlarged image of the slit with the shadow of the string across it, on a distant screen, or on the ground-glass of a camera (with the lens removed) placed in front of it. The principal difficulty which remains to be overcome is that with the low frequency of the fork employed, the diameter of the wire used cannot be made very small, and the magnified image of its shadow across the slit is excessively broad. This cannot be conveniently avoided by the use of fine wires and a high frequency fork, as the amplitude of vibration then becomes too small for successful projection. The device finally adopted by me is to flatten a very short length of the wire just opposite the illuminated slit, with a hammer, so as to have the wire at this point practically a very thin flat ribbon seen edgewise and therefore giving a very sharp and narrow shadow on the screen. As the linear density of the wire is even locally altered by this treatment, the vibrational modes are quite unaffected, and it becomes possible to obtain satisfactory curves on the screen by bowing the string and setting the fork in vibration.

An extraordinary and most interesting variety of curves can be demonstrated to a large audience in this way, and the gradual rotation of the curve due to slight imperfections in the tuning has a very striking and almost stereoscopic effect.

To obtain photographic records of these curves, the arra-

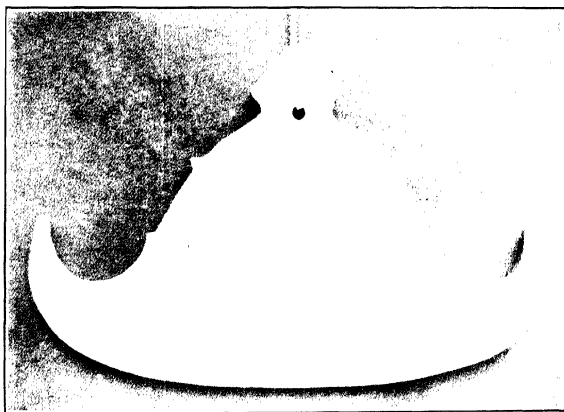
ment described above is adopted. The string is so tuned that when it is bowed, the curve seen on the ground-glass of the camera is practically steady, and a picture is secured with an exposure of only $\frac{1}{100}$ th of a second. With such an exposure, the effect of any imperfection in the tuning on the sharpness of the figure photographed is quite negligible. The available light is so great that, even with this short exposure, it is generally found necessary to use a neutral-tinted screen in front of the shutter of the camera to prevent over-exposure of the plate.

Plate XVI shows nine pictures secured in the preliminary work. In each case, it will be seen that the frequency of the fundamental vibration of the string employed was double that of the fork employed, and some of the curves are evidently identical with those observed and drawn by Helmholtz. I hope later to find time to secure and publish a more extensive series of curves, grouped according to the places of bowing and observation and with various ratios of frequency between the vibrations of the bowed string and that of the interruptor-fork.

In concluding this Bulletin, I have great pleasure in acknowledging the valuable assistance I have received from Mr. Asutosh Dey, the senior demonstrator of the Association Laboratory, in planning and carrying out the experiments described above. I must also express my gratitude to Dr. Amrita Lal Sircar who, as Honorary Secretary, put the resources of the Laboratory unreservedly at my disposal during hours at which few institutions, if any, would remain open for work, and who was also unfailing with his personal encouragement and advice.

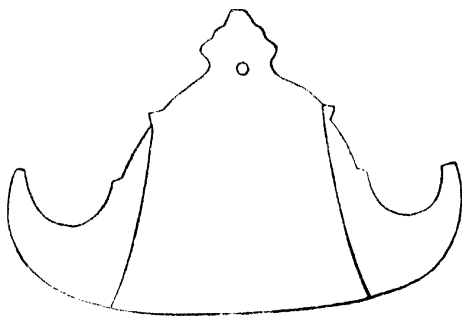
some recent experiments in which I endeavoured to analyse by the aid of resonators, the acoustical effect due to the rotation of 'double' source of sound about a line passing through it perpendicular to its axis. A 'double' source of sound is, in effect, a combination of two equal but opposite sources of identical frequency situated indefinitely near each other, the product of their intensity into their distance being a finite quantity. A 'double' source emits no sound in its equatorial plane, i.e. in any direction at right angles to its axis. It is therefore readily seen that if a double source is rotated with uniform angular velocity about any line passing through it and lying in its equatorial plane, an observer situated at any distance from the source would perceive the effect of 'beats' simulated in a most remarkable manner. At each half-revolution, as the equatorial plane of the source sweeps through his position, there would be an extinction of the sound followed during the intervals by a recrudescence, and the frequency of the 'beats' is therefore twice the number of rotations made per second.

It is not at all a difficult matter to secure in experiment what is very approximately equivalent to a 'double' source of sound. It is known (Rayleigh's 'Theory of Sound,' Art. 325) that the disturbance due to the vibration of a sphere as a rigid body is the same as that corresponding to a double source at the centre whose axis coincides with the line of the sphere's vibration. In practice, the flexural vibrations of a small flat disc or plate would be a very fair substitute, provided that one of the vibrating segments of the plate emits a sufficiently powerful and steady sound which is not sensibly interfered with by any effect of the other segments. With a circular plate, for example, the mode of vibration with one nodal circle would be the most suitable for the purpose, and this would be improved, if the thickness of the disc varied in such manner from the centre outwards that the nodal circle is rather nearer the edge of the disc than the theoretical position for a disc of uniform thickness. I have found that the gong of bell-metal shown in Plate IV serves most admirably as a double source of sound. When struck with a wooden hammer, its vibrations in the gravest mode are powerfully excited and a clear steady note is emitted which continues to be audible for over a minute, free from



Vibrating Plate used as a Double Source of Sound.

any 'beats' or quavers (so long as the plate is held in position) and free from perceptible admixture with upper partials, these latter ceasing to be audible within two or three seconds of the impact of the hammer. This very desirable result appears to be a consequence of the variation of the thickness of the plate, which is least at the centre of the horizontal edge of the plate and gradually increases in all directions upwards. The 'wings' at the end of the base of the plate are particularly massive and appear to be both ornamental and useful, as can be seen from the position of the nodal lines shown in the figure. The emission of sound is principally due to the central segment of the plate, the two wings having very little effect as could be shown by covering up the former when the bulk of the sound falls off.



When suspended by a fairly long thin cord and struck on one of the wings, the gong is excited and can at the same time be set in rotation at any moderate speed. The effect of 'beats' is then reproduced in a most remarkable manner. The analysis of these beats was effected in the following manner by the aid of a tuning fork mounted on a resonance box. The frequency of the fork was 512, and was a little greater than that of the vibrations of the plate. By loading the prongs of the fork suitably with wax, it was found possible to get the two to be of identical frequency or to have the frequency of the fork slightly in excess or defect as desired. For the analysis of the beats due to the rotation, the fork mounted upon its box was itself used as a resonator.

Experiment 1.

The plate was excited and held in position. The fork was adjusted for exact unison, and held a few feet off. On stopping the vibration of the plate the fork was found to be sounding loudly.

Experiment 2.

As above, but with the plate in rotation. There was no resonance.

Experiments 3 and 4.

The fork was thrown out of unison in one direction or the other by putting on or removing a little wax, so that when sounded it made approximately *one* beat per second with the gong. On repeating Exp. 1, there was no resonance.

Experiments 5 and 6.

As in experiments 3 and 4 but the plate in rotation at such a speed that *two* beats per second were heard. There was marked resonance.

These observations are of interest as illustrating the mechanical production and analysis of 'true' beats. At each half-revolution of the plate we get an extinction of the sound, followed by a revival in which the phase of the motion is reversed. This can readily be understood from the fact that the motion on the two sides of the plate is in any instant in opposite phases, a compression on one side and a rarefaction on the other and vice versa.

SOME RECIPROCAL EFFECTS.

On adjusting the fork and the vibrating plate used in the preceding experiments to *exact* unison and sounding the former, very marked resonance was exhibited by the plate. The arrangement was found to be very sensitive, a deviation from unison giving only one beat per second practically abolishing the resonance. This observation suggested the following experiments which appear to be of some interest as illustrations of the dynamical theory of resonance.

Experiment 7.

Fork and plate in exact unison. Holding the fork mounted on its box a few feet from and *facing* the plate, it was found on exciting the fork that the plate showed marked resonance.

Experiment 8.

As above, but with the fork and resonance box in *the plane* of the plate. Resonance extremely feeble.

Experiment 9.

As in experiment 7, but with the plate in rotation. No resonance.

Experiments 10 and 11.

As in the experiment 7 with the plate held in position, but the fork thrown slightly out of unison in one direction or the other (to the extent of one beat per second) by putting on or removing a little wax. No resonance.

Experiments 12 and 12(a).

As in experiments 10 and 11, but with the plate in rotation at a speed of one revolution per second in either case. Marked resonance was noticed.

MATHEMATICAL NOTE.

The equivalence of the effect at any point due to a rotating double source and that due to the beats of two simple tones of differing frequencies can easily be shown theoretically. Thus:

The velocity-potential of a double source is proportional to the real part in

$$\frac{d}{dx} \left[\frac{e^{ikh(at-r)}}{r} \right] = \mu \frac{d}{dr} \left[\frac{e^{ikh(at-r)}}{r} \right] = -ikh \mu \frac{e^{ikh(at-r)}}{r} \left[1 + \frac{1}{ikr} \right]$$

where x is the axis and μ is the cosine of the angle between x and r . Putting $\mu = \cos \theta = \cos \omega t$ where ω is the angular velocity of rotation of the double source, the expression given above may be written as

$$-ikh e^{-ikr} \frac{e^{i(ha+\omega)t} + e^{i(ha-\omega)t}}{2r} \left[1 + \frac{1}{ikr} \right].$$

From this, the inference is obvious.

